Mathematics Revision Notes 2009
RI(JC) Mathematics Society

Pure Mathematics	**Page**

Statistics	**Page**

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Functions

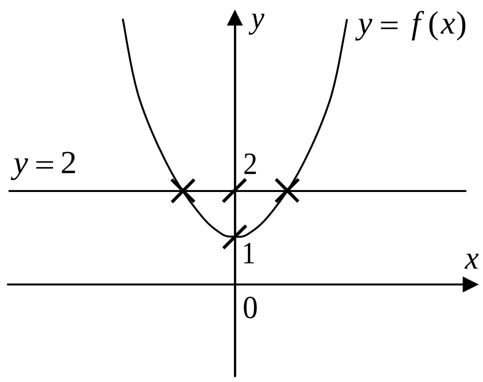
A **function** f is a relation which maps each element in its **domain** D_f to one and only one element in its **range** R_f, e.g. $f: x \mapsto x^2+1,\ x \in \mathbb{R}$, where x^2+1 is the **rule** of f and $\mathbb{R}$ is the domain of f. Use the **Vertical Line Test** to determine if a given relation f is a function.

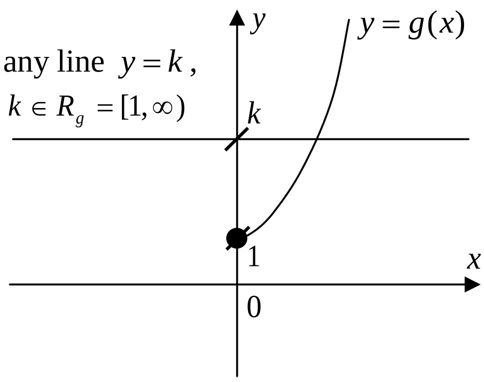
Replace D_f and R_f below with the actual domain and range of f in interval notation.

<u>One-one Functions</u>

Use the **Horizontal Line Test** to determine if a given function is one-one.

e.g. $f: x \mapsto x^2+1,\ x \in \mathbb{R}$ and $g: x \mapsto x^2+1,\ x \geq 0$.





Since **the** horizontal line $y=2$, where $2 \in R_f = [1,\infty)$, **does not cut** the graph of f at one and only one point, f is not one-one.

Since **any** horizontal line $y=k$, where $k \in R_g = [1,\infty)$, **cuts** the graph of g at one and only one point, g is one-one.

OR

Since $f(-1) = f(1) = 2$,
where $-1, 1 \in D_f = \mathbb{R}$, f is not one-one.

g is a **restriction** of f if f and g have the same rule and $D_g \subseteq D_f$, i.e. g is f with its **domain restricted**. Even if f is not one-one, a restriction of f which is one-one can be defined. Use **turning points** or **points where f is undefined** to show that f is not one-one or restrict its domain.

Inverse Functions

f^{-1}, the **inverse function** of f, is a function such that for all $x \in D_f$ and $y \in R_f$, $f^{-1}(y) = x$ if and only if $f(x) = y$.

- f^{-1} exists if and only if f is one-one.
- Domain of $f^{-1} = R_f$
- Range of $f^{-1} = D_f$

To find the **rule of f^{-1}** given the **rule of f**:

1. Let $y = f(x)$.
2. Express x in terms of y, such that $x = g(y)$, where $g(y)$ is a function of y.
3. Then $f^{-1}(x) = g(x)$.

The rules of ff^{-1} and $f^{-1}f$ are always x, but they usually have **different domains**.
$D_{ff^{-1}} = D_{f^{-1}}$, but $D_{f^{-1}f} = D_f$.

Composite Functions

gf is the **composite function** of f followed by g.

- gf exists if and only if $R_f \subseteq D_g$.
- Domain of $gf = D_f$
- Range of $gf = R_g$, with the domain of g restricted to R_f.

FAQ

Q: Sketch the graph of f.
A: Substitute the rule of f, then sketch the graph using GC.

Q: What is the domain/range of f?
A: Observe from the rule of f by substituting values, or from the graph using GC.

Q: What is the minimum value of k such that f, with its domain restricted to (k, ∞), is one-one?
A: Find the minimum value of k such that f, with its domain restricted, passes the Horizontal Line Test. Usually, $x = k$ will be a **turning point** or **point where f is undefined** on the graph of f.

Q: Solve $f(x) = f^{-1}(x)$.
A: Solve $f(x) = x$ or $f^{-1}(x) = x$ instead, since the graphs of f and f^{-1} must intersect on the line $y = x$. **(Example 2)**

Q: **Solve $ff^{-1}(x) = f^{-1}f(x)$.**

A: The solution set is the intersection of the domains of f and f^{-1}, since both ff^{-1} and $f^{-1}f$ have rule x.

Q: **Sketch the graph of ff^{-1} (or $f^{-1}f$).**

A: The graph of ff^{-1} is the section of the line $y = x$, where $x \in D_{f^{-1}} = R_f$. Similarly, the graph of $f^{-1}f$ is the section of the line $y = x$, where $x \in D_f$.

Q: **Sketch the graphs of $f(x)$ and $f^{-1}(x)$ (on a single diagram).**

A: Show their geometrical relationship: the graphs of f and f^{-1} are reflections of each other about the line $y = x$.

<u>**Example 1**</u> [NJC06/CT/Q9 (Modified)]

A mapping f is given by $f : x \mapsto \dfrac{1}{(x-1)^2} - 1, x \in \mathbb{R}$.

State the largest possible domain of f in the form $(-\infty, b)$, where $b \in \mathbb{R}$, such that the inverse function of f exists. Hence define f^{-1} in a similar form.

The functions g and h are defined as follows:

$$g : x \mapsto \ln(x+2), x \in (-1, 1)$$
$$h : x \mapsto x^2 - 2x - 1, x \in \mathbb{R}^+$$

Determine whether the composite function gh exists.
Give the rule and domain of the composite function hg and find its range.

	Solution	**Comments**
(a)	Largest possible $D_f = (-\infty, 1)$ ■	f^{-1} exists if and only if f is one-one. Observe a discontinuity at $x = 1$ by sketching the graph of f on the GC or from $\dfrac{1}{(x-1)^2}$. Restrict D_f based on the discontinuity instead of looking for turning points.
	$D_{f^{-1}} = R_f = (-1, \infty)$	Use the GC to find R_f.

Let $y = f(x) = \dfrac{1}{(x-1)^2} - 1,\ x \in (-\infty, 1)$.

$$y + 1 = \frac{1}{(x-1)^2}$$

$$(x-1)^2 = \frac{1}{y+1}$$

$$x - 1 = \pm\frac{1}{\sqrt{y+1}}$$

$$x = 1 \pm \frac{1}{\sqrt{y+1}}$$

$$x = 1 - \frac{1}{\sqrt{y+1}}$$

or $1 + \dfrac{1}{\sqrt{y+1}}$ (rejected)

Use D_f to decide which of the 2 expressions for x to accept and which to reject. Here, $D_f = (-\infty, 1)$, so $x < 1$, and $\dfrac{1}{\sqrt{y+1}} > 0$ for all $y \in \mathbb{R}$.

$f^{-1} : x \mapsto 1 - \dfrac{1}{\sqrt{x+1}},\ x \in (-1, \infty)$ ■

Express f^{-1} in a similar form, and write down both its rule and domain.

(b)(i) $h(x) = x^2 - 2x - 1 = (x-1)^2 - 2$

$R_h = (h(1), \infty) = (-2, \infty)$

$h(x)$ has a minimum point at $x = 1$, and its domain $\mathbb{R}^+$ includes 1.

Since $(-2,\infty) \not\subseteq (-1,1)$, $R_h \not\subseteq D_g$, hence the composite function gh does not exist. ■

gh exists if and only if $R_h \subseteq D_g$.

(b)(ii) $D_{hg} = D_g = (-1,1)$

$hg(x) = h(g(x)) = h(\ln(x+2))$

$hg : x \mapsto (\ln(x+2))^2 - 2\ln(x+2) - 1$, $x \in (-1,1)$ ■

You do not need to show hg exists, since the question already assumes it.

$R_g = (g(-1), g(1)) = (0, \ln 3)$

$g(x)$ is strictly increasing on $(-1,1)$.

$R_{hg} = R_h$ with D_h restricted to R_g

$R_{hg} = [h(1), h(0)) = [-2, -1)$ ■

In the restricted $D_h = R_g = (0, \ln 3)$, $h(x)$ decreases from 0 to 1, reaches its minimum at $h(1) = -2$, then increases from 1 to $\ln 3$. Since $h(0) > h(\ln 3)$, $h(0)$ is the larger endpoint of R_{hg}.

<u>Example 2</u> [HCI06/Promo/Q11 (Modified)]

The function f is defined by

$$f : x \mapsto \frac{1}{1+e^x}, x \geq 0$$

By considering the domains of ff^{-1} and $f^{-1}f$, solve $ff^{-1}(x) = f^{-1}f(x)$.
Show the graphs of $f(x)$, $f^{-1}(x)$ and $ff^{-1}(x)$ on a single sketch.

Solution

(i)

$$D_{ff^{-1}} = D_{f^{-1}} = R_f = \left(0, \frac{1}{2}\right]$$

$$D_{f^{-1}f} = D_f = [0, \infty)$$

For $ff^{-1}(x) = f^{-1}f(x)$,

$$x \in \left(0, \frac{1}{2}\right] \cap [0, \infty) = \left(0, \frac{1}{2}\right]$$

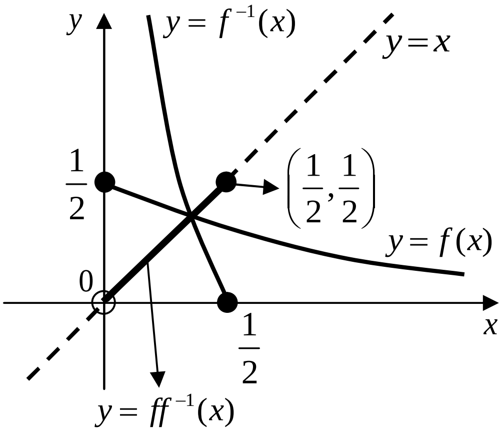
$$0 < x \leq \frac{1}{2}$$ ■

Comments

The rules of ff^{-1} and $f^{-1}f$ are always x, but they may have different domains.

Hence, $ff^{-1}(x) = f^{-1}f(x)$ if and only if x is in the domain of both ff^{-1} and $f^{-1}f$, i.e.:
$x \in D_{ff^{-1}} \cap D_{f^{-1}f} = R_f \cap D_f$

(ii)



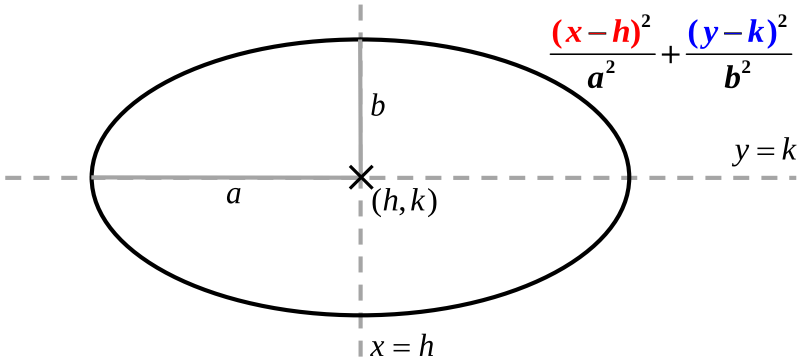
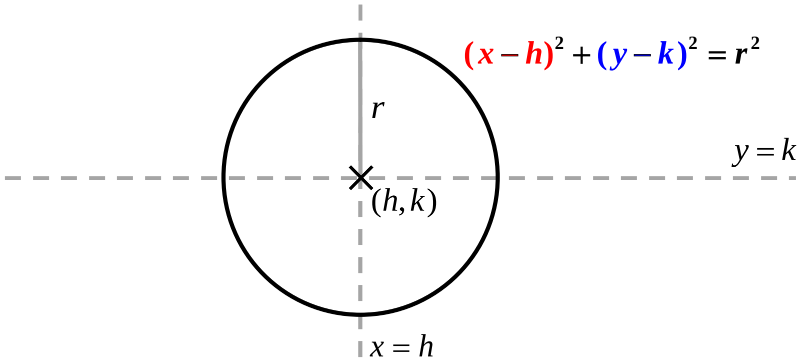
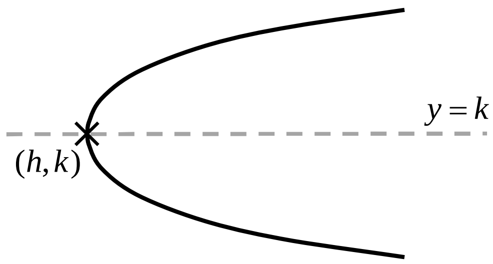
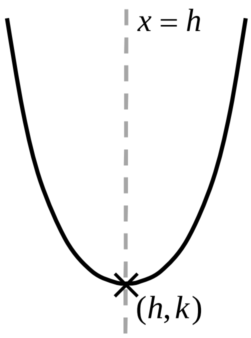
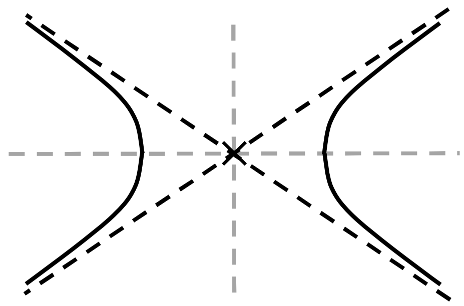
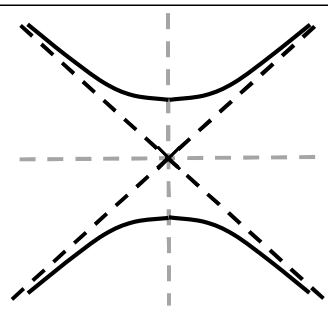
Sketch the graphs on the GC first. When the rule of $f^{-1}(x)$ is unknown, key in *DrawInv(*Y_1*)* to draw the inverse of the graph of Y_1.

<u>GC syntax:</u>
Set $Y_1 = 1/(1+e\wedge(X))$, then:
[2ND][PRGM] → 8:DrawInv
→ [VARS] → Y-VARS → Function… → Y_1
→ [ENTER]

Graphing Techniques

Graphs of Conic Sections

The dotted lines represent **lines of symmetry** or **asymptotes**, not the axes.

Ellipse	$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$ 	
Circle	$(x-h)^2+(y-k)^2=r^2$ 	
Parabola	$x-h=b(y-k)^2$ 	$y-k=b(x-h)^2$ 
Hyperbola	 $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$	 $\frac{(y-k)^2}{b^2}-\frac{(x-h)^2}{a^2}=1$

Properties of Hyperbolas

For both forms of hyperbolas above,

- Centre: (h,k)
- Horizontal line of symmetry: $x=h$
- Vertical line of symmetry: $y=k$
- Oblique asymptote (sloping upwards): $y-k=\frac{b}{a}(x-h)$
- Oblique asymptote (sloping downwards): $y-k=-\frac{b}{a}(x-h)$

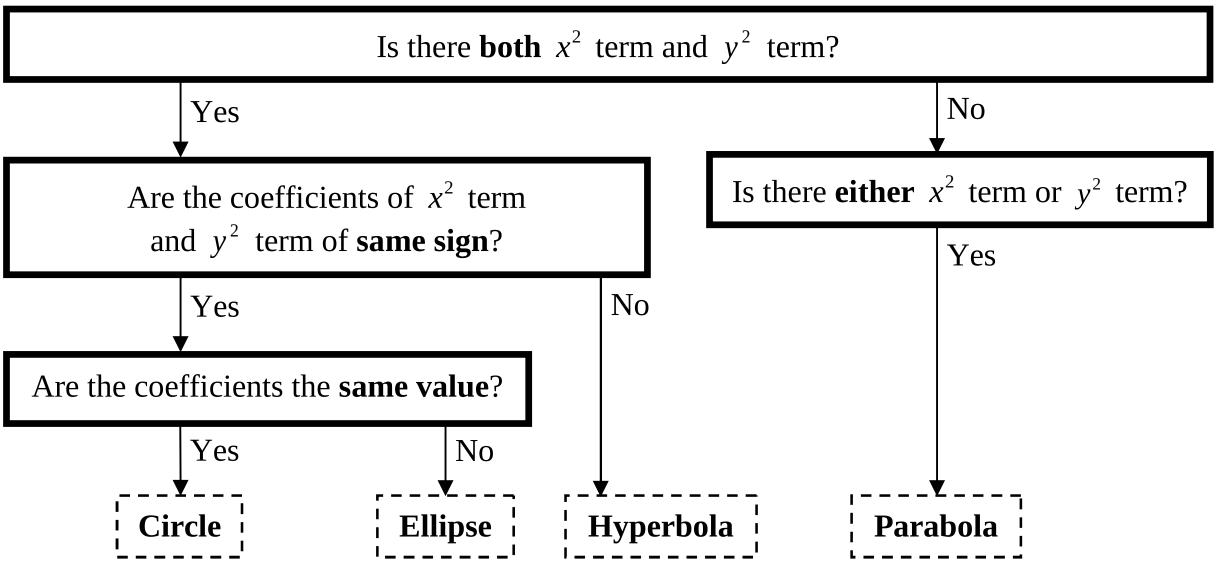
If the $\boldsymbol{x}$ **term** comes first in the equation of a hyperbola, i.e. $\frac{(\mathbf{x}-h)^2}{a^2}-\frac{(\mathbf{y}-k)^2}{b^2}=1$, then the graph looks like a **handwritten** $\boldsymbol{x}$ (with left/right parts) and always intersects the $\boldsymbol{x}$ **axis**.

Sketching Graphs of Conic Sections

- There is a special case for each conic where its centre or vertex is at the **origin** (i.e. $h=0$, $k=0$). To draw a conic with centre/vertex (h,k), **translate** this special case by replacing x with $x-h$ and y with $y-k$.

- If the equation of a conic is given in the general form $\boldsymbol{Ax^2+By^2+Cx+Dy+E=0}$, complete the square to convert it into the standard forms given in the table above.

- When the equation of a conic is given in general form, to identify the conic:



Graphs of Rational Functions

$\boldsymbol{y = \frac{ax+b}{cx+d}}$, where $c \neq 0$, numerator and denominator have **no common factor**.

- Using **long division**, express y as $p + \frac{q}{cx+d}$, where p and q are constants.
- Vertical asymptote: $x = -\frac{d}{c}$
  - When $cx + d = 0$, $\frac{q}{cx+d}$ is undefined, hence y is undefined.
- Horizontal asymptote: $y = p = \frac{a}{c}$
  - As $x \to \pm\infty$, $\frac{q}{cx+d} \to 0$, hence $y \to p$.

$\boldsymbol{y = \frac{ax^2+bx+c}{dx+e}}$, where $a, d \neq 0$, numerator and denominator have no common factor.

- Using **long division**, express y as $px + q + \frac{r}{dx+e}$, where p, q, and r are constants.
- Vertical asymptote: $x = -\frac{e}{d}$
  - When $dx + e = 0$, $\frac{r}{dx+e}$ is undefined, hence y is undefined.
- Oblique asymptote: $y = px + q$
  - As $x \to \pm\infty$, $\frac{r}{dx+e} \to 0$, hence $y \to px + q$.

Let the initial graph be of $y = f(x)$, and let a be a positive constant.

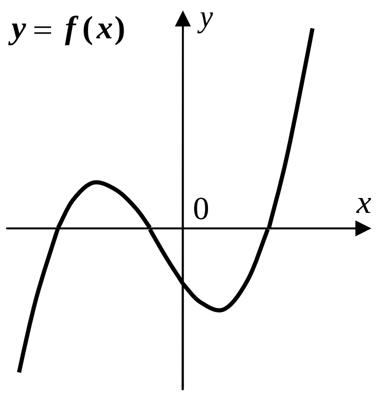
Translating Graphs

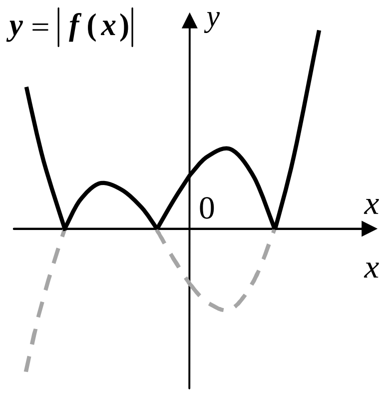
Translate by a units in the:	**In the equation**	**Equation after transformation**
positive y-direction	Replace y by $y - a$	$y - a = f(x)$ i.e. $y = f(x) + a$
negative y-direction	Replace y by $y + a$	$y + a = f(x)$ i.e. $y = f(x) - a$
positive x-direction	Replace x by $x - a$	$y = f(x - a)$
negative x-direction	Replace x by $x + a$	$y = f(x + a)$

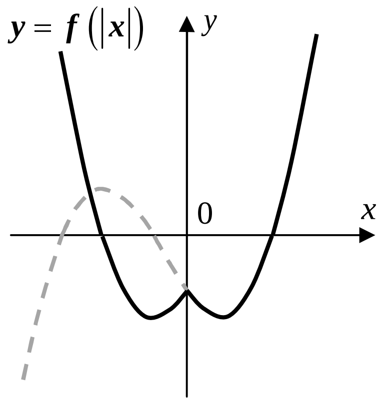
Stretching & Reflecting Graphs

Stretch parallel to the y-axis by factor a, with x-axis invariant	Replace y by $\frac{y}{a}$	$\frac{y}{a} = f(x)$ i.e. $y = af(x)$
Stretch parallel to the x-axis by factor a, with y-axis invariant	Replace x by $\frac{x}{a}$	$y = f\left(\frac{x}{a}\right)$
Reflect in the x-axis ($y = 0$)	Replace y by $-y$	$-y = f(x)$ i.e. $y = -f(x)$
Reflect in the y-axis ($x = 0$)	Replace x by $-x$	$y = f(-x)$

Transformations Involving Modulus







Reflect the part of the graph of $y = f(x)$ where $f(x) < 0$ in the x-axis.

Delete the part of the graph of $y = f(x)$ where $x < 0$.

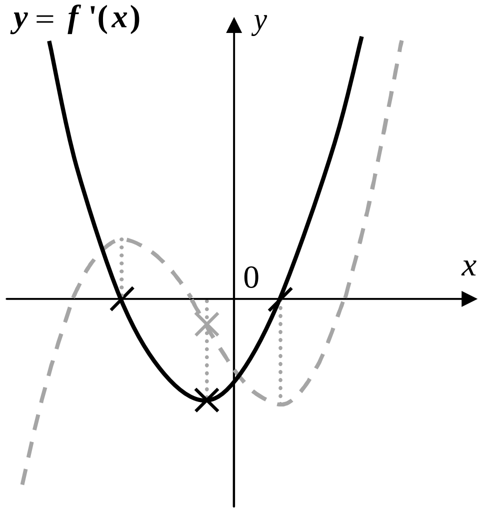
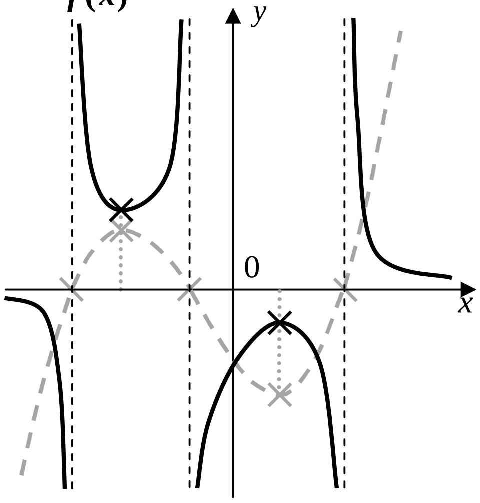
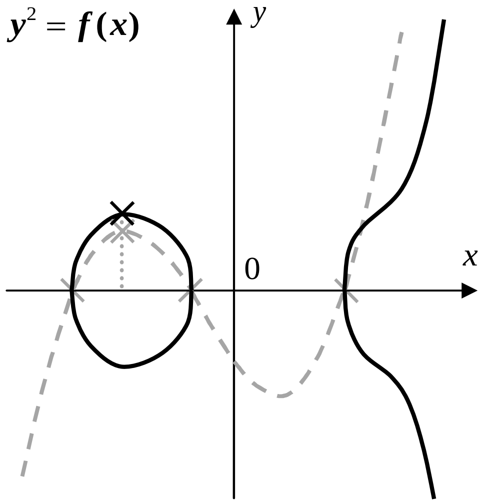
Reflect the part of the graph of $y = f(x)$ where $x > 0$ in the y-axis.

Other Transformations

Assume the same graph of $y = f(x)$ as above, which is drawn with a **dashed line** below.

Other transformations include:

- $y = f'(x)$
- $y = \frac{1}{f(x)}$
- $y^2 = f(x)$
    - Related to $y = \sqrt{f(x)}$ or $y = -\sqrt{f(x)}$

<table>
<tr><th>Graph</th><th>Graph of $y = f(x)$</th><th>Transformed graph</th></tr>
<tr><td>$y = f'(x)$
</td><td>Stationary point
Increasing
Decreasing
Concave upwards
Concave downwards
Maximum gradient
Minimum gradient</td><td>x-intercept
Positive
Negative
Increasing
Decreasing
Maximum point
Minimum point</td></tr>
<tr><td>$y = \frac{1}{f(x)}$
</td><td>Positive
Negative
Increasing
Decreasing
Approaches ∞ or $-\infty$
Approaches 0
Maximum point
Minimum point</td><td>Positive
Negative
Decreasing
Increasing
Approaches 0
Approaches ∞ or $-\infty$
Minimum point
Maximum point</td></tr>
<tr><td>$y^2 = f(x)$
</td><td colspan="2">1. Discard the part of $y = f(x)$ where $f(x) < 0$.
2. Look at x-intercepts of $y = f(x)$ to decide the kind of tangent on the transformed graph.
• **1** distinct real root: **vertical** tangent
• **2** equal real roots: **sharp** tangent
• $>$**2** equal real roots: **horizontal** tangent
3. This is the graph of $y = \sqrt{f(x)}$.
4. Find the graph of $y = -\sqrt{f(x)}$ by reflecting the graph of $y = \sqrt{f(x)}$ in the x-axis.
5. Combine the graphs of $y = \sqrt{f(x)}$ and $y = -\sqrt{f(x)}$ to get the graph of $y^2 = f(x)$.</td></tr>
</table>

Sketching a Transformed Graph

1. Work from the left of the graph ($x = -\infty$) to the right ($x = \infty$).
2. Identify critical features on the original graph.
3. On the new graph, label all corresponding coordinates of points, equations of asymptotes, stationary points etc. which are given on the original graph.
   - For points, write ', e.g. use A' to denote point A after transformation.
4. Check your answer.
   - Use substitutions, e.g. when $f(x) = 10$, $y = \dfrac{1}{f(x)} = 0.1$.
   - Use GC to graph normally or use its *Conics* application.

Sequences of Transformations Involving Modulus

Example: Transform the graph of $y = f(x)$ to that of $y = f\left(2 - 3|x|\right)$

Solution: $f(x) \to f(2+x) \to f(2-x) \to f(2-3x) \to f\left(2-3|x|\right)$

When $|x|$ is involved, replace x by $|x|$ **last**, to get a graph symmetric about the y-axis.

Example: Transform the graph of $y = f(x)$ to that of $y = f\left(|2-3x|\right)$

Solution: $f(x) \to f\left(|x|\right) \to f\left(|2+x|\right) \to f\left(|2-x|\right) \to f\left(|2-3x|\right)$

Here $2-3x$ is inside the modulus, hence replace x by $|x|$ **first**, then transform $|x|$ to $|2x-3|$.

General Tips

- Generally, perform transformations from the innermost to outermost brackets.
- Write the exact phrasing of the transformations when the question asks you to "describe" them, e.g. write "translate by 3 units…", not "translate for 3 units…".

Example [YJC/I/Q8]

The curve C has equation

$$y = \frac{x^2 + kx + 1}{2x+3}, k \in \mathbb{R}^+$$

(i) Obtain the equations of the asymptotes of C.
(ii) Find the value of k for which the x-axis is tangent to C.
(iii) Sketch the case for $k = 3$. Hence, using a graphical method, find the range of values of x that satisfy the inequality $\left|x^2 + 3x + 1\right| > |4x+6|$.

Solution / **Comments**

(i) $y=\dfrac{x^2+kx+1}{2x+3}$

$=\dfrac{x}{2}+\dfrac{k}{2}-\dfrac{3}{4}+\dfrac{13-6k}{4(2x+3)}$

Use long division to get a quotient of $\dfrac{x}{2}+\dfrac{k}{2}-\dfrac{3}{4}$ and a remainder of $\dfrac{13-6k}{4}$.

Equations of asymptotes:
$x=-\dfrac{3}{2}$ and $y=\dfrac{x}{2}+\dfrac{k}{2}-\dfrac{3}{4}$. ■

Graphs of $y=\dfrac{ax^2+bx+c}{dx+e}$ usually have 2 asymptotes:

1. To find the **vertical** asymptote, set the x-term in the denominator $=0$, i.e. $2x+3=0$.
2. To find the **oblique** asymptote, take $y\to\pm\infty$ such that $\dfrac{13-6k}{4(2x+3)}\to 0$ and $y\to\dfrac{x}{2}+\dfrac{k}{2}-\dfrac{3}{4}$.

(ii) Where x-axis is tangent to C,
$y=0\Rightarrow x^2+kx+1=0$

Since there can only be 1 value of x where $y=0$,
$D=k^2-4(1)(1)=0$
$k^2=4$
$k=2$ or -2 (rejected $\because k\in\mathbb{R}^+$)
$k=2$ ■

For a quadratic equation $ax^2+bx+c=0$, $a\neq 0$, discriminant $D=b^2-4ac$.

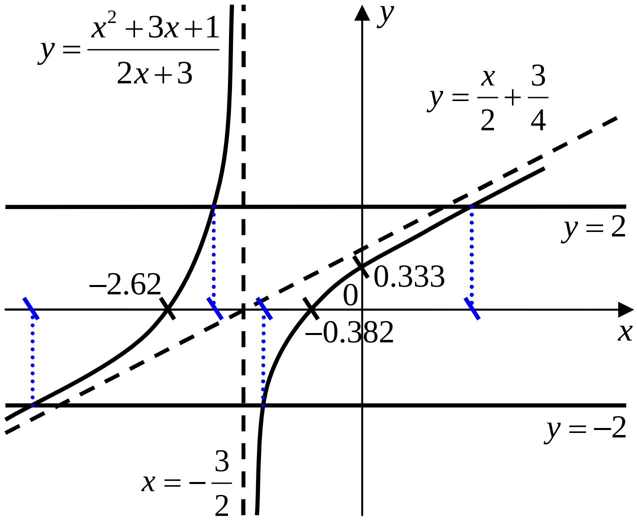
$D>0$: 2 distinct real roots.
$D=0$: 1 (repeated) real root.
$D<0$: No real roots.

Always try using the discriminant with quadratic equations in Graphing Techniques questions.

Differentiation is tedious and will result in a complicated expression. Furthermore, if the question states "algebraically", differentiation cannot be used.

Use the GC to plot the graph for $k=2$ to check your answer.

(iii)



When $k = 3$, $y = \frac{x^2+3x+1}{2x+3}$.

$|x^2+3x+1| > |4x+6|$

$|x^2+3x+1| > 2|2x+3|$

$\frac{|x^2+3x+1|}{|2x+3|} = \left|\frac{x^2+3x+1}{2x+3}\right| > 2$

$|y| > 2$

$y < -2$ or $y > 2$

From the graph,

$x < -5.79$ or $-1.79 < x < -\frac{3}{2}$

or $-\frac{3}{2} < x < -1.21$ or $x > 2.79$ ■

Substitute $k = 3$ into the equations of the graph and its asymptotes, and sketch it using the GC. Label all axis intercepts (to 3 s.f.) and asymptotes.

"Graphical method" suggests:

1. Manipulate given inequality until 1 side, say LHS, resembles equation of graph.
2. Sketch graph of RHS on GC and your diagram.
3. The solution usually involves points of intersection of the 2 graphs.
   - e.g. the **blue marks** on the x -axis in the diagram above.

Exclude the values of x for which the function is undefined, e.g. $x = \frac{3}{2}$ above.

We can divide both sides of the inequality by $|2x+3|$ because it is always positive.

Equations & Inequalities

General Approach

1. Zero the inequality (bring all the terms to one side, such that the other side is 0).
2. Combine the terms to form a single fraction.
3. Factorize the terms as far as possible.
   - Use $\frac{f(x)}{g(x)} \le 0 \Leftrightarrow f(x)g(x) \le 0$ to convert the inequality to a product of factors. Since $\left[g(x)\right]^2$ is always positive, multiplying it on both sides preserves the inequality sign.
4. Eliminate factors which are either always positive or always negative.
   - Complete the square, e.g. for all $x \in \mathbb{R}$, $x^2 - 4x + 5 = (x-2)^2 + 1 \ge 1 > 0$.
5. Draw number line and graph and shade the regions where the inequality holds.
6. Write the range of values of x (e.g. $1 < x \le 2$).
   - To write the **set of values**, use set-builder notation, NOT interval notation.
     - e.g. $\{x \in \mathbb{R} : 1 < x \le 2\}$, NOT $(1, 2]$
   - Check for any values (especially endpoints) to be excluded
     - e.g. $x = 2$ when the inequality contains $\frac{1}{x-2}$
     - e.g. whether the inequality is strict ($<$) or not ($\le$).

Using the GC

- Find intersection between 2 graphs: [2ND][TRACE] → 5:intersect
  - Find roots of a graph: [2ND][TRACE] → 2:zero
- Find stationary points of a graph: [2ND][TRACE] → 3:minimum (or 4:maximum)
- Solve system of linear equations: [APPS] → 5:PlySmlt2

FAQ

- Solve a given inequality, using algebraic or graphical approaches.
- Using the solution to an original inequality, solve a new inequality.
  - Substitute x with some function of x, $f(x)$ (usually $-x$, $\frac{1}{x}$, $\ln x$, e^x, etc.) in the original inequality, such that it becomes the **new** inequality. Then, substitute x with the same function $f(x)$ in the original solution set to obtain the **new** solution set.
- Model a problem with a system of linear equations and solve the system.

General Tips

- In general, **do not cross-multiply** inequalities! Cross-multiplying only works if the denominators on both sides of the inequality are either both positive or both negative.

- Use the GC if the question does not specify "exact" values. But be careful when using the GC directly: some solution intervals are not visible at certain zoom levels! **(Example 1)**

- Given an inequality involving modulus, e.g. $|f(x)| \leq a$ where a is a positive constant,
  - Divide it into cases, where $f(x) < 0$ and $f(x) \geq 0$.
    - $|f(x)| \leq a \Rightarrow -a \leq f(x) \leq a$
    - $|f(x)| \geq a \Rightarrow f(x) \leq -a$ or $f(x) \geq a$
  - **OR** Square both sides to get $[f(x)]^2 \leq a^2$, removing the modulus.

- Remember to flip the sign of the inequality when you:
  - Multiply or divide both sides by a negative number
    - e.g. $a \leq b \Rightarrow -2a \geq -2b$
  - Take the reciprocal of both sides, if they are either both positive or both negative
    - e.g. $a \leq b \Rightarrow \frac{1}{a} \geq \frac{1}{b}$, where $a, b > 0$ (or $a, b < 0$)

- Apply your numerical answer to the context of the question, if appropriate.

Example [N2006/II/Q1]

Solve the inequality $\frac{x-9}{x^2-9} \leq 1$.

Solution	**Comments**
$\frac{x-9}{x^2-9} \leq 1$	Use the **General Approach** outlined above.
$\frac{x-9}{x^2-9} - 1 \leq 0$	1. Zero the inequality.

$$\frac{x-9-\left(x^2-9\right)}{x^2-9} \le 0$$

$$\frac{x-x^2}{x^2-9} \le 0$$

2. Combine the terms to form a single fraction.

$$\left(x-x^2\right)\left(x^2-9\right) \le 0$$

$$x(1-x)(x+3)(x-3) \le 0$$

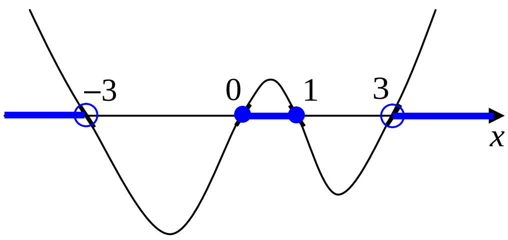
3. Factorize the terms as far as possible.

$$x(x-1)(x+3)(x-3) \ge 0$$

Make the coefficient of the highest power of x, here x^4, positive, so that the graph in Step 5 will start as **positive from the right**.

Remember to flip the sign of the inequality when multiplying both sides by -1.

Skip Step 4, since the terms are already linear.



5. Draw number line and graph and shade the regions where inequality holds.

- Mark the roots -3, 0, 1, 3 of the equation $y = x(x-1)(x+3)(x-3)$.
- The graph has x-intercepts at the roots.
- If $f(x)$ is a polynomial with the coefficient of the highest power of x being positive, the graph of $f(x)$ will start as **positive from the right**.
- The graph has a "wavy" shape.

$x < -3$ or $0 \le x \le 1$ or $x > 3$ ■

Write the range of values of x, checking for any values (especially endpoints) to be excluded.

Since the original inequality has denominator $x^2 - 9 = (x+3)(x-3)$, exclude $x = -3$ and $x = 3$ where the inequality is undefined.

Do NOT use commas: e.g. $x < -3$, $0 \le x \le 1$, $x > 3$. Write out "or" between every 2 adjacent inequalities.

Alternative method from this step

(Sketch the graph of $y=\frac{x-9}{x^2-9}-1$*)*

From the GC,
$x<-3$, $0\le x\le 1$, or $x>3$

Alternative method

Since the question doesn't ask for exact values, zero the inequality, then use the GC directly.

Always **zoom in** to check places where the graph is **near the** x**-axis**, where the inequality is near equality.

e.g. the solution interval $0\le x\le 1$ is not clearly visible at some zoom levels, e.g. *ZStandard* and *ZDecimal*.

Arithmetic & Geometric Progressions

Below, **AP** denotes **arithmetic progression** and **GP** denotes **geometric progression**.

For an **AP** $u_1, u_2, u_3, \ldots$ with first term $a = u_1$ and common difference d :

- nth term, $u_n = a + (n-1)d$
- Sum of first n terms, $S_n = \frac{n}{2}\left[2a + (n-1)d\right] = \frac{n}{2}(a + u_n)$
- If a, b, c form an AP, then $b - a = c - b \Leftrightarrow 2b = a + c$

For a **GP** $u_1, u_2, u_3, \ldots$ with first term $a = u_1$ and common ratio r:

- nth term, $u_n = ar^{n-1}$
- Sum of first n terms, $S_n = \frac{a(1-r^n)}{1-r} = \frac{a(r^n - 1)}{r-1}$
- Sum to infinity, $S_\infty = \frac{a}{1-r}$ (S_∞ exists only if $|r| < 1$)
- If a, b, c form a GP, then $\frac{b}{a} = \frac{c}{b} \Leftrightarrow b^2 = ac$

For **any sequence** $u_1, u_2, u_3, \ldots$

- $u_k = S_k - S_{k-1}$, where $k > 0$.
- $u_k + u_{k+1} + \ldots + u_m = S_m - S_{k-1}$, where $m \geq k > 0$.

FAQ

Q: Find various properties of an AP or GP, given other properties of it.

A:
1. Write down a, d (or r), n, any kth term u_k, any sum of first k terms S_k which is given in the question.
2. Use the various AP & GP formulae to solve.

Q: Show that a sequence is an AP.

A: Show that the difference is common, i.e. $u_k - u_{k-1}$ is constant for all k.

Q: Show that a sequence is a GP.

A: Show that the ratio is common, i.e. $\frac{u_k}{u_{k-1}}$ is constant for all k.

Q: Model and solve a problem using an AP or GP.

A:
1. Identify if the sequence given is an AP or GP.
2. Write down a, d (or r), n, any kth term u_k, any sum of first k terms S_k which is given in the question.
3. Use the various AP & GP formulae to solve.

Q: Find the least/most n such that some condition holds.

A:
1. Form an inequality in terms of n.
2. Solve manually **OR** use GC's table of values/graph.

Example 1 [JJC/II/Q1]

An arithmetic progression of positive terms is such that twice the sum of the first nine terms is equal to the sum of the next nine terms. Let T_n denote the nth term of the arithmetic progression. Given that T_1, 20, T_{16} forms a geometric progression, find the first term and the common difference of the arithmetic progression.

Solution	**Comments**
Let the arithmetic progression have first term a, common difference d, and sum of first n terms S_n.	Define all variables that will be used. Interpret the question in the context of the defined variables: find a and d.
$2S_9 = S_{18} - S_9$ $3S_9 = S_{18}$	Always write sums of consecutive terms in terms of differences of sums of first k terms: $T_{10} + T_{11} + \ldots + T_{18} = S_{18} - S_9$
$3\times\frac{9}{2}\left(2a+8d\right) = \frac{18}{2}(2a+17d)$ $27a+108d = 18a+153d$ $a = 5d$	Simplify the first sentence in the question to an equation in terms of a and d, which we want to find.
$\frac{20}{T_1} = \frac{T_{16}}{20} \Rightarrow T_1T_{16} = 400$ $a\left(a+15d\right) = 400$	Simplify the other statement in the question to another equation in terms of a and d.
$d = \frac{a}{5}$ $a\left(a+3a\right) = 4a^2 = 400$ $a^2 = 100$	Now substitute the first equation, $a = 5d$, into the second equation. Simultaneous equations are a common feature in AP & GP questions.

$a = 10$ or -10 (rejected $\because a > 0$ as the arithmetic progression is of positive terms)
$a = 10$, $d = 2$ ■

Look for conditions given in the question whenever you must choose a correct value.

Example 2 [N97/I/Q15 (Part)]

A bank has an account for investors. Interest is added to the account at the end of each year at a fixed rate of 5% of the amount in the account at the beginning of that year. A man decides to invest $\$x$ at the beginning of one year and then a further $\$x$ at the beginning of the second and each subsequent year. He also decides that he will not draw any money out of the account, but just leave it, and any interest, to build up.

(i) Show that, at the end of n years, when the interest for the last year has been added, he will have a total of $\$21\left(1.05^n - 1\right)x$ in his account.

(ii) After how many complete years will he have, for the first time, at least $\$12x$ in his account?

Solution

(i) Let u_n be the amount of money in \$ in the man's account at the end of n years.

$u_1 = 1.05x$

$u_2 = 1.05(1.05x + x) = \left(1.05^2 + 1.05\right)x$

$u_3 = 1.05\left(\left(1.05^2 + 1.05\right)x\right) + x$

$= \left(1.05^3 + 1.05^2 + 1.05\right)x$

$\vdots$

$u_n = \left(1.05^n + 1.05^{n-1} + \ldots + 1.05\right)x$

$= \dfrac{1.05\left(1.05^{n-1} - 1\right)}{1.05 - 1}x$

$= 21\left(1.05^{n-1} - 1\right)x$ ■ (shown)

(ii) $u_n \geq 12x$

$21\left(1.05^{n-1} - 1\right)x \geq 12x$

$21\left(1.05^{n-1} - 1\right) \geq 12$ ($\because x > 0$)

$1.05^{n-1} \geq \dfrac{12}{21} + 1 = \dfrac{11}{7}$

Comments

Define u_n as this to avoid writing the same phrase repeatedly. Remember to specify the units (in \$).

Evaluate successive values of u_k, i.e. u_1, u_2, u_3, and observe a pattern for u_n. Here, "replace" 3 with n.

Do not simplify 1.05^2, 1.05^3 etc., or the pattern will be difficult to observe.

Apply the formula for the sum of the first n terms of a GP to simplify u_n.

"At least" means use $\geq$ instead of $>$.

Manipulate the inequality to solve for n.

$$(n-1)\ln 1.05 \ge \ln\left(\frac{11}{7}\right)$$

$$n \ge \frac{\ln\left(\frac{11}{7}\right)}{\ln 1.05} = 9.26 \text{ (3 s.f.)}$$

$\therefore$ least integer value of $n = 10$ ■

When n is in the exponent, take ln on both sides. Here, the sign of the inequality of preserved when dividing both sides by $\ln 1.05 > 0$.

Switch the sign when dividing both sides by $\ln x$, where $0 < x < 1$.

<u>Example 3</u> [VJC06/J1CT/Q9b (Modified)]

At the beginning of the year, Xin Fu deposited $100,000 with a bank that pays 10% interest per annum at the end of each year. After the interest is credited, he immediately withdraws $12,000. Likewise, Xin Fu will again withdraw $12,000 at the end of each subsequent year, immediately after the bank's interest has been credited.

Express Xin Fu's bank account balance in $ after his nth withdrawal in terms of n.

Solution	**Comments**
Let u_n be Xin Fu's bank account balance in $ after his nth withdrawal.	Define u_n clearly, and specify the units ($).
$u_1 = 1.1(100000) - 12000$ $u_2 = 1.1u_1 - 12000$ $= 1.1^2(100000) - 12000(1.1+1)$ $u_3 = 1.1u_2 - 12000$ $= 1.1^3(100000) - 12000(1.1^2+1.1+1)$ $\vdots$ $u_n = 1.1^n(100000)$ $-12000(1.1^{n-1}+1.1^{n-2}+\ldots+1)$	Evaluate successive values of u_k, i.e. u_1, u_2, u_3, and observe a pattern for u_n. Here, "replace" 3 with n, 2 with $n-1$, etc. Do not simplify 1.1^2, 1.1^3 etc., or the pattern will be difficult to observe.
$= 1.1^n(100000) - 12000\left[\frac{1(1.1^n-1)}{1.1-1}\right]$ $= 1.1^n(100000) - 120000(1.1^n-1)$ $= 120000 - 20000(1.1^n)$ $= 20000(6-1.1^n)$ ■	Apply the formula for the sum of the first n terms of a GP to simplify u_n. Count the number of terms carefully. Here, $(1.1^{n-1}+1.1^{n-2}+\ldots+1)$ is a sum of powers of 1.1 ranging from 0 to $n-1$, so it has n terms (not $n-1$).

<u>Alternative methods</u>

Write the recurrence relation:

$$u_n = 1.1u_{n-1} - 12000$$

Now,

- Use **repeated substitution**.
- **OR** Use **induction**, especially if the result to be proven: $u_n = 20000\left(6 - 1.1^n\right)$ is already given.

Refer to **<u>Example 2</u>** in *Summation, Recurrence & Induction.*

Summation, Recurrence & Induction

Summation

- **Definition:** $\sum_{r=m}^{n} f(r) = f(m) + f(m+1) + \ldots + f(n)$, where $m \le n$.
- For constants a and b, functions of r $f(r)$ and $g(r)$ and $m \le n$,
  - $\sum_{r=m}^{n} af(r) = a\sum_{r=m}^{n} f(r)$
  - $\sum_{r=m}^{n} \left[af(r) \pm bg(r)\right] = a\sum_{r=m}^{n} f(r) \pm b\sum_{r=m}^{n} g(r)$
  - Sum of AP: general term has the form $a+br$.
  - Sum of GP: general term has the form ab^r.
  - $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$
  - $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$
  - $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2 = \left(\frac{1}{2}n(n+1)\right)^2 = \left(\sum_{r=1}^{n} r\right)^2$
- Convert sums with $r \ne 1$ to use those formulas above showing r starting **from 1**.
  - $\sum_{r=m}^{n} f(r) = \sum_{r=1}^{n} f(r) - \sum_{r=1}^{m-1} f(r)$
- Use **method of difference** to find $\sum u_r$ when consecutive terms cancel out. **<u>(Example 1)</u>**

Recurrence

- For a sequence $u_1, u_2, \ldots$ a **recurrence relation** defines it recursively. Each term u_n is defined as a function of the preceding terms $u_1, u_2, \ldots, u_{n-1}$.
- AP can be expressed as $u_n = u_{n-1} + d$, where the common difference d is a constant.
- GP can be expressed as $u_n = ru_{n-1}$, where the common difference r is a constant.

Mathematical Induction

Step 1: Base case. Aim: Show that P_1 is true.

Let P_n be the statement "…" When $n=1$, L.H.S. = … R.H.S. = … L.H.S. = R.H.S. $\therefore P_1$ is true.

Step 2: Inductive step. Aim: Show that P_{k+1} is true whenever P_k is true.

Assume P_k is true for some $k \in \mathbb{Z}^+$, i.e. … To prove P_{k+1} is true, i.e. … Now, L.H.S. $=$ … $=$ R.H.S. i.e. P_{k+1} is true whenever P_k is true. Since P_1 is true, by Mathematical Induction, P_n is true for all $n \in \mathbb{Z}^+$.

- The assumption that P_k is true for some $k \in \mathbb{Z}^+$ is the **induction hypothesis**.
- Make the necessary changes if the question specifies a base value of n **other than 1**.
- Know where you're going when performing the inductive step:
    1. Convert L.H.S. of P_{k+1} to involve L.H.S. of P_k.
    2. Since we assume P_k is true, substitute L.H.S. of P_k with R.H.S. of P_k.
        - The induction hypothesis **must be used** somewhere in the induction.
    3. Convert the expression to R.H.S. of P_{k+1} (proven).

<u>Example 1</u> [RJC08/Assignment 7/Q4 (Modified)]

Express $\dfrac{2r}{(r+1)(r+2)(r+3)}$ in partial fractions.

Hence, find the sum of the first n terms of the series

$$\frac{2}{2\times3\times4}+\frac{4}{3\times4\times5}+\frac{6}{4\times5\times6}+\dots$$

Find the sum to infinity of this series.

Solution

Let

$$\frac{2r}{(r+1)(r+2)(r+3)}=\frac{A}{r+1}+\frac{B}{r+2}+\frac{C}{r+3}$$

for some constants A,B,C.

$$2r = A(r+2)(r+3)+B(r+1)(r+3)$$
$$+C(r+1)(r+2)$$

Comments

Recall the partial fractions decomposition for linear factors in the denominator (in MF15). Substitute appropriate values of r to find A,B,C.

Check that the solution allows for cancelling out terms using method of difference in the later part: $-1+4-3=0$.

When $r=-1$, $-2=2A \Rightarrow A=-1$
When $r=-2$, $-4=-B \Rightarrow B=4$
When $r=-3$, $-6=2C \Rightarrow C=-3$

$$\frac{2r}{(r+1)(r+2)(r+3)}$$
$$=-\frac{1}{r+1}+\frac{4}{r+2}-\frac{3}{r+3} \quad \blacksquare$$

Required sum $=\sum_{r=1}^{n}\frac{2r}{(r+1)(r+2)(r+3)}$

Observe that the terms of the given series are of the form $\frac{2r}{(r+1)(r+2)(r+3)}$, where $r=1,2,\ldots$.

$$=\sum_{r=1}^{n}\left(-\frac{1}{r+1}+\frac{4}{r+2}-\frac{3}{r+3}\right)$$
$$=-\frac{1}{2}+\frac{4}{3}-\frac{3}{4}$$
$$-\frac{1}{3}+\frac{4}{4}-\frac{3}{5}$$
$$-\frac{1}{4}+\frac{4}{5}-\frac{3}{6}$$
$$\vdots$$
$$-\frac{1}{n-1}+\frac{4}{n}-\frac{3}{n+1}$$
$$-\frac{1}{n}+\frac{4}{n+1}-\frac{3}{n+2}$$
$$-\frac{1}{n+1}+\frac{4}{n+2}-\frac{3}{n+3}$$

If a sum cannot be found directly or you have decomposed the term into partial fractions, think of the method of difference.

Write out the lines where $r=1, 2, 3$ at the top, and $r=n-2, n-1, n$ at the bottom. Always do this for the first 3 values and last 3 values of r.

The terms should be in terms of n, not r.

Show the cancelling out of terms clearly, leaving the fractions in their original form (do not simplify them).

$$=-\frac{1}{2}+\frac{4}{3}-\frac{1}{3}-\frac{3}{n+2}+\frac{4}{n+2}-\frac{3}{n+3}$$
$$=\frac{1}{2}+\frac{1}{n+2}-\frac{3}{n+3} \quad \blacksquare$$

Write out the remaining terms and simplify.

Sum to infinity $=\sum_{r=1}^{\infty}\left(-\frac{1}{r+1}+\frac{4}{r+2}-\frac{3}{r+3}\right)$

$$=\lim_{n\to\infty}\left[\sum_{r=1}^{n}\left(-\frac{1}{r+1}+\frac{4}{r+2}-\frac{3}{r+3}\right)\right]$$
$$=\lim_{n\to\infty}\left(\frac{1}{2}+\frac{1}{n+2}-\frac{3}{n+3}\right)$$
$$=\frac{1}{2} \quad \blacksquare$$

The sum to infinity is the limit as $n\to\infty$ of the sum to n terms. As $n\to\infty$, $\frac{1}{n+2}\to 0$ and $\frac{3}{n+3}\to 0$. Hence, the sum to infinity is $\frac{1}{2}$.

Example 2 (Refer to first paragraph of **Example 3** in *Arithmetic & Geometric Progressions*.)

Let u_n be Xin Fu's bank account balance in $ after his nth withdrawal.

(i) State an expression for u_n in terms of u_{n-1}.
(ii) Hence, find u_n in terms of n.
(iii) Prove your result in part (ii) for all $n \geq 1$ by mathematical induction.

	Solution	**Comments**
(i)	$u_n = 1.1u_{n-1} - 12000$ ■	
(ii)	$u_1 = 1.1(100000) - 12000 = 98000$	Use **repeated substitution**. The recurrence relation expresses u_n in terms of u_{n-1}. Use the relation repeatedly to express u_n in terms of u_{n-2}, u_{n-3} etc.
	$u_n = 1.1u_{n-1} - 12000$ $= 1.1(1.1u_{n-2} - 12000) - 12000$ $= \mathbf{1.1^2 u_{n-2}} - 12000(\mathbf{1.1} + 1)$ $= 1.1^2(1.1u_{n-3} - 12000) - 12000(1.1+1)$ $= \mathbf{1.1^3 u_{n-3}} - 12000(\mathbf{1.1^2} + 1.1 + 1)$ $\vdots$ $= \mathbf{1.1^{n-1} u_1} - 12000(\mathbf{1.1^{n-2}} + 1.1^{n-3} + \ldots + 1)$	Observe a pattern and express u_n in terms of u_1 (which can be evaluated). Here, the general form for u_n is: $\mathbf{1.1^k u_{n-k}} - 12000(\mathbf{1.1^{k-1}} + \ldots + 1)$ $u_{n-k} = u_1 \Rightarrow n - k = 1 \Rightarrow k = n - 1$. To express u_n in terms of u_1, substitute $k = n-1$ into the general form.
	$= 1.1^{n-1}(98000) - 12000\left[\dfrac{1(1.1^{n-1} - 1)}{1.1 - 1}\right]$ $= 1.1^{n-1}(98000) - 120000(1.1^{n-1} - 1)$ $= 120000 - 22000(1.1^{n-1})$ $= 120000 - 20000(1.1^n)$ $= 20000(6 - 1.1^n)$ ■	$(1.1^{n-2} + 1.1^{n-3} + \ldots + 1)$ is a sum of powers of 1.1 ranging from 0 to $n-2$, so it has $n-1$ terms (not $n-2$).
(iii)	Let P_n be the statement "$u_n = 20000(6 - 1.1^n)$".	

When $n=1$,
L.H.S. $=u_1=98000$
R.H.S. $=20000\left(6-1.1^1\right)=98000$
L.H.S. $=$ R.H.S.
$\therefore P_1$ is true.

Show how the value 98000 is calculated.

Assume P_k is true for some $k\in\mathbb{Z}^+$,
i.e. $u_k=20000\left(6-1.1^k\right)$.

Assume the **induction hypothesis**.
Replace n by k in the statement of P_n.

To prove P_{k+1} is true,
i.e. $u_{k+1}=20000\left(6-1.1^{k+1}\right)$,

Replace n by $k+1$ in the statement of P_n.

Now, L.H.S. $=u_{k+1}=1.1u_k-12000$

Use the given recurrence relation, $u_{k+1}=1.1u_k-12000$, to convert L.H.S. of P_{k+1} to involve L.H.S. of P_k.

$=1.1\left[20000\left(6-1.1^k\right)\right]-12000$
(by induction hypothesis)
$=132000-12000-20000\left(1.1^{k+1}\right)$
$=120000-20000\left(1.1^{k+1}\right)$
$=20000\left(6-1.1^{k+1}\right)$
$=$ R.H.S. (proven)

Use the induction hypothesis that P_k is true, i.e. $u_k=20000\left(6-1.1^k\right)$, to substitute L.H.S. of P_k with R.H.S. of P_k.

Convert the expression to the R.H.S. of P_{k+1}. Manipulate the expression to be more and more similar to R.H.S. of P_{k+1}.

i.e. P_{k+1} is true whenever P_k is true.

Since P_1 is true, by Mathematical Induction, P_n is true for all $n\in\mathbb{Z}^+$.

<u>Example 3</u> [N01/I/13b]

Use induction to prove that

$$3(1!)+7(2!)+13(3!)+\ldots+\left(n^2+n+1\right)(n!)=(n+1)^2(n!)-1$$

Solution

$3(1!)+7(2!)+13(3!)+\ldots+\left(n^2+n+1\right)(n!)$

$=\sum_{r=1}^{n}\left(r^2+r+1\right)(r!)$

Comments

Rewrite the L.H.S. of the given equation as a sum, for easier manipulation.

$\left(n^2+n+1\right)(n!)$ shows the general form. Check that it works for the first 3 terms given.

Let P_n be the statement

"$\sum_{r=1}^{n}\left(r^2+r+1\right)(r!)=(n+1)^2(n!)-1$".

When $n=1$,

L.H.S. $=\left(1^2+1+1\right)(1!)=3$

R.H.S. $=(1+1)^2(1!)-1=3$

L.H.S. = R.H.S.

$\therefore P_1$ is true.

Show how the value 3 is calculated.

Assume P_k is true for some $k\in\mathbb{Z}^+$,

i.e. $\sum_{r=1}^{k}\left(r^2+r+1\right)(r!)=(k+1)^2(k!)-1$.

Assume the **induction hypothesis**.

Replace n by k in the statement of P_n.

To prove P_{k+1} is true,

i.e. $\sum_{r=1}^{k+1}\left(r^2+r+1\right)(r!)=(k+2)^2\left[(k+1)!\right]-1$,

Replace n by $k+1$ in the statement of P_n.

Now, L.H.S. $=\sum_{r=1}^{k+1}\left(r^2+r+1\right)(r!)$

$=\sum_{r=1}^{k}\left(r^2+r+1\right)(r!)$

$\quad+\left[(k+1)^2+(k+1)+1\right]\left[(k+1)!\right]$

$=\sum_{r=1}^{k}\left(r^2+r+1\right)(r!)+\left(k^2+3k+3\right)\left[(k+1)!\right]$

Convert L.H.S. of P_{k+1} to involve L.H.S. of P_k.

When L.H.S. of P_n involves a sum of n terms, extract the ($k+1$)th term from the sum of ($k+1$) terms in L.H.S. of P_{k+1} to obtain the sum of k terms in L.H.S. of P_k.

$=(k+1)^2(k!)-1+\left(k^2+3k+3\right)\left[(k+1)!\right]$

Use the induction hypothesis that P_k is true to substitute L.H.S. of P_k with R.H.S. of P_k.

$= (k+1)\left[(k+1)!\right] - 1$
$+ (k^2 + 3k + 3)[(k+1)!]$
$= (k^2 + 4k + 4)\left[(k+1)!\right] - 1$
$= (k+2)^2\left[(k+1)!\right] - 1$
$=$ R.H.S. (proven)

i.e. P_{k+1} is true whenever P_k is true.

Since P_1 is true, by Mathematical Induction, P_n is true for all $n \in \mathbb{Z}^+$.

Convert the expression to the R.H.S. of P_{k+1}. Manipulate the expression to be more and more similar to R.H.S. of P_{k+1}.

$(k+1)!$ is the product of all positive integers up to $(k+1)$, and $k!$ is the product up to k. Hence:

$$(k+1)! = (k+1)(k!)$$
$$\frac{(k+1)!}{k!} = k+1$$

In general:

$$m! = \left[m(m-1)(m-2)\ldots(k+1)\right](k!)$$
$$\frac{m!}{k!} = m(m-1)(m-2)\ldots(k+1)$$

where $m \geq k \geq 0$.

Differentiation

Basic Formulae

Let u and v be functions of x. Let y be a function of u.

$$\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$$

$$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$$ (Product Rule)

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$ (Quotient Rule)

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$ (Chain Rule)

Derivatives of Standard Functions

Let n be an integer and a be a constant.

Unless otherwise stated, the derivatives are valid for all $x \in \mathbb{R}$.

NOT in MF15:

$y = f(x)$	$\frac{dy}{dx} = f'(x)$	**Range of validity/Comments**
x^n	nx^{n-1}	Memorize $\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$ and $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$.
e^x	e^x	
a^x	$a^x \ln a$	
$\ln x$	$\frac{1}{x}$	$x > 0$
$\sin x$	$\cos x$	
$\cos x$	$-\sin x$	
$\tan x$	$\sec^2 x$	
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$	
$\cot x$	$-\operatorname{cosec}^2 x$	

In MF15:

$y = f(x)$	$\frac{dy}{dx} = f'(x)$	**Range of validity/Comments**
$\sec x$	$\sec x \tan x$	
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$	$\lvert x \rvert < 1$
$\cos^{-1} x$	$-\frac{1}{\sqrt{1-x^2}}$	$\lvert x \rvert < 1$
$\tan^{-1} x$	$\frac{1}{1+x^2}$	

- $\sin^{-1} x \neq \frac{1}{\sin x}$, while $\operatorname{cosec} x = \frac{1}{\sin x}$.
- $\cos^{-1} x \neq \frac{1}{\cos x}$, while $\sec x = \frac{1}{\cos x}$.
- $\tan^{-1} x \neq \frac{1}{\tan x}$, while $\cot x = \frac{1}{\tan x}$.

<u>Techniques of Differentiation</u>

- Let $f(x)$ be any function and $f'(x)$ be its derivative given in the table above. To find the derivative of $f(g(x))$, use the **Chain Rule** with $u = g(x)$ and $y = f(g(x)) = f(u)$: **<u>(Example 1)</u>**

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$\frac{d}{dx} f(g(x)) = f'(g(x)) \times g'(x)$$

- **Implicit differentiation:** If an equation involving x and y can be expressed as $y = f(x)$, differentiating explicitly gives $\frac{dy}{dx} = \frac{d}{dx} f(x)$. However, if the equation cannot be easily expressed as $y = f(x)$, we can find $\frac{dy}{dx}$ by differentiating the equation term-by-term, with respect to x. **<u>(Example 2)</u>**

  Use the **Chain Rule** to differentiate a function of y, say $g(y)$, with respect to x.

$$\frac{d}{dx} g(y) = \frac{d}{dy} g(y) \times \frac{dy}{dx}$$

- **Parametric differentiation:** When a curve involving x and y is described by equations of the form $x = f(t)$ and $y = g(t)$, with t in some given range of values, each value of the parameter t gives a point with coordinates $(x, y) = \left(f(t), g(t)\right)$ on the curve. **<u>(Example 3)</u>**

  - Use the **Chain Rule** to find $\frac{dy}{dx}$, after calculating $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$$

<u>Applications of Differentiation</u>

- **Properties of $f(x)$ from its Derivatives**

For a range of values $a < x < b$, $f(x)$ is:	**Only if for all $a < x < b$:**	**i.e. Gradient ($\frac{dy}{dx}$) is:**
Strictly increasing	$\frac{dy}{dx} > 0$	Positive
Strictly decreasing	$\frac{dy}{dx} < 0$	Negative
Concave upwards	$\frac{d^2y}{dx^2} > 0$	Increasing
Concave downwards	$\frac{d^2y}{dx^2} < 0$	Decreasing

  - If for a range of values $a < x < b$, the graph of $f(x)$ looks like a **<u>U</u>** ($\cup$) or a part of it, then $f(x)$ is concave **<u>U</u>**pwards. If it looks like an n ($\cap$) or a part of it, then $f(x)$ is concave downwards.

- **First Derivative Test:** To determine the nature of a stationary point at $x = \alpha$, evaluate $\frac{dy}{dx}$ at a x value slightly less than α and a x value slightly more than α, and observe whether these values of $\frac{dy}{dx}$ are positive or negative.
  - Use it when $\frac{d^2y}{dx^2}$ is difficult to find ($\frac{dy}{dx}$ is difficult to differentiate) or $\frac{d^2y}{dx^2} = 0$.
  - In the following table, write the **actual values** of α^-, α, α^+ and $\frac{dy}{dx}$ at these values of x.

x	α^-	α	α^+	α^-	α	α^+	α^-	α	α^+	α^-	α	α^+
$\frac{dy}{dx}$	>0	$=0$	<0	<0	$=0$	>0	>0	$=0$	>0	<0	$=0$	<0
Graph	/	—	\	\	—	/	/	—	/	\	—	\
Nature of stationary point	Maximum turning point			Minimum turning point			Stationary point of inflexion					

- **Second Derivative Test:** To determine the nature of a stationary point at $x=\alpha$, evaluate $\frac{d^2y}{dx^2}$ at $x=\alpha$ and observe whether it is positive or negative.
    - In the following table, write the **actual values** of α and $\frac{d^2y}{dx^2}$ at $x=\alpha$.

$\frac{d^2y}{dx^2}$ **at** $x=\alpha$	<0	>0	$=0$
Nature of stationary point	Maximum turning point	Minimum turning point	No conclusion possible

- **Tangents & Normals**

Q: Find the equation of a straight line.
A: Given the gradient m and a point on the line with coordinates (x_1, y_1),

$$y - y_1 = m(x - x_1)$$

Q: Find the gradient, m, of a straight line.
A: Given 2 points on the line with coordinates (x_1, y_1) and (x_2, y_2),

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Given that the line is the **tangent** to a curve $y = f(x)$ at $x=\alpha$,

$$m = \frac{dy}{dx}\text{, where } x=\alpha$$

Given that the line is the **normal** to a curve $y = f(x)$ at $x=\alpha$,

$$m = -1 \div \frac{dy}{dx}\text{, where } x=\alpha$$

since $m_1 m_2 = -1$, where m_1 and m_2 are the gradients of 2 perpendicular lines.

Q: Find the axis intercepts of a curve.

A: x-intercepts: Let $y=0$ in the equation of the curve and solve for x.
y-intercepts: Let $x=0$ in the equation of the curve and solve for y.

Q: Find the value of x at which the tangent is horizontal (parallel to the x-axis).

A:
Let $\frac{dy}{dx}=0$ and solve for x.

Q: Find the value of x at which the tangent is vertical (parallel to the y-axis).

A:
Let $\frac{dx}{dy}=1\div\frac{dy}{dx}=0$ and solve for x.

Q: Find the distance, d, between 2 points with coordinates (x_1, y_1) and (x_2, y_2).

A:
$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

- **Maximization, Minimization & Rates of Change**

  - Always show that a stationary point you have found is indeed a maximum point (or minimum point), using either **Derivative Test** to determine its nature.
  - Remember to include **units** in your final answer.
  - Make sure that there is **only 1 variable** in the expression before differentiating the expression with respect to that same variable.
  - Use the **Chain Rule** to connect rates of change.
  - **Similar triangles** and **trigonometric ratios** are useful in forming expressions.

<u>Example 1</u>

Differentiate a^x with respect to x.

Solution	**Comments**
$a^x = e^{\ln a^x}$	(since $k=e^{\ln k}$ for all $k\in\mathbb{R}$)
$= e^{x\ln a}$	(since $\log_a b^r = r\log_a b$)

Let $u = x\ln a$, $y = e^u$.

$$\frac{d}{dx}a^x = \frac{d}{dx}e^{x\ln a} = \frac{dy}{dx}$$

By the Chain Rule,

$$\frac{dy}{dx} = \frac{dy}{du}\times\frac{du}{dx} = \frac{d}{du}e^u \times \frac{d}{dx}x\ln a = \left(e^u\right)\left(\ln a\right)$$

$$= e^{x\ln a}\left(\ln a\right) = a^x \ln a \ \blacksquare$$

The details are unnecessary for actual working. Only write:

$$\frac{d}{dx}a^x = \frac{d}{dx}e^{x\ln a} = e^{x\ln a}\ln a = a^x \ln a \ \blacksquare$$

Substitute u back with $x\ln a$.

<u>Example 2</u> [NJC06/Midyear]

The equation of a curve C is given by $y^3 + yx^2 + 2x^3 = k$, where k is a constant. Find $\frac{dy}{dx}$ in terms of x and y.

Solution

$$y^3 + yx^2 + 2x^3 = k$$

Differentiating with respect to x,

$$3y^2\frac{dy}{dx} + x^2\frac{dy}{dx} + 2xy + 6x^2 = 0$$

$$\left(3y^2 + x^2\right)\frac{dy}{dx} + 2xy + 6x^2 = 0$$

$$\left(3y^2 + x^2\right)\frac{dy}{dx} = -\left(2xy + 6x^2\right)$$

$$\frac{dy}{dx} = -\frac{2xy + 6x^2}{3y^2 + x^2} = -\frac{2x\left(y + 3x\right)}{3y^2 + x^2} \ \blacksquare$$

Comments

Whenever an equation involving x and y cannot be easily expressed as $y = f\left(x\right)$, use implicit differentiation.

Using the Chain Rule,

$$\frac{d}{dx}y^3 = \frac{d}{dy}y^3 \times \frac{dy}{dx} = 3y^2\frac{dy}{dx}$$

Using the Product Rule,

$$\frac{d}{dx}yx^2 = x^2\frac{d}{dx}y + y\frac{d}{dx}x^2 = x^2\frac{dy}{dx} + 2xy$$

Group the terms containing $\frac{dy}{dx}$ by factoring it out, then shift $\frac{dy}{dx}$ to one side and factorize further.

Example 3 [SRJC08/Prelim/I/9i]

The parametric equations of a curve are given by

$$x=\frac{1}{4}\left(e^{2t}-2t\right) \text{ and } y=e^{t}\text{ , where } 0\le t\le 1.$$

Find $\frac{dy}{dx}$ and show that $\frac{d^2y}{dx^2}<0$ for $0<t\le 1$.

Solution

$$\frac{dy}{dt}=e^{t}$$
$$\frac{dx}{dt}=\frac{1}{4}e^{2t}(2)-\frac{1}{2}=\frac{1}{2}(e^{2t}-1)$$

Comments: Whenever equations for x and y are given in terms of a parameter t, use parametric differentiation.

Find $\frac{dy}{dt}$ and $\frac{dx}{dt}$ first.

$$\frac{dy}{dx}=\frac{dy}{dt}\div\frac{dx}{dt}$$
$$=\frac{2e^{t}}{e^{2t}-1} \blacksquare$$

Comments: Use the Chain Rule.

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{2e^{t}}{e^{2t}-1}\right)$$

Comments: $\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)$

$$=\frac{d}{dt}\left(\frac{2e^{t}}{e^{2t}-1}\right)\times\frac{dt}{dx}$$

Comments: Use the Chain Rule to differentiate the expression involving t with respect to t.

$$=2\frac{d}{dt}\left(\frac{e^{t}}{e^{2t}-1}\right)\times\frac{1}{\left(\frac{dx}{dt}\right)}$$

Comments: $\frac{dt}{dx}=\frac{1}{\left(\frac{dx}{dt}\right)}$

$$=2\left[\frac{\left(e^{2t}-1\right)e^{t}-e^{t}\left(2e^{2t}\right)}{\left(e^{2t}-1\right)^{2}}\right]\left(\frac{2}{e^{2t}-1}\right)$$
$$=\left[\frac{4}{\left(e^{2t}-1\right)^{3}}\right]\left(-e^{2t}-1\right)e^{t}$$
$$=-\frac{4e^{t}\left(e^{2t}+1\right)}{\left(e^{2t}-1\right)^{3}}$$

For all $t \in \mathbb{R}$, $0 < t \le 1$,
$e^t > 1$ and $e^{2t} > 1$.

Try to find an inequality satisfied by terms in $\frac{d^2y}{dx^2}$ to prove that $\frac{d^2y}{dx^2} < 0$ for $0 < t \le 1$. Use the given range of values of t.

Since $e^t > 1 > 0$,
$e^{2t} + 1 > 2 > 0$ and
$e^{2t} - 1 > 0$,

$$\frac{d^2y}{dx^2} = (-4)\left[\frac{e^t\left(e^{2t}+1\right)}{\left(e^{2t}-1\right)^3}\right] < 0 \ \blacksquare \ \text{(shown)}$$

For all $x \in \mathbb{R}$, $x > 0$, $e^x > 1$.

Show that the factors in the numerator and denominator of $\frac{e^t\left(e^{2t}+1\right)}{\left(e^{2t}-1\right)^3}$ are all positive, hence the whole fraction must be positive.

<u>Example 4</u> [N1998/I/7]

A spherical balloon is being inflated and, at the instant when its radius is 3 m, its surface area is increasing at the rate of 2 m^2 s^{-1}. Find the rate of increase, at the same instant, of

(i) the radius,
(ii) the volume.

[The formulae for the surface area and volume of a sphere are $A = 4\pi r^2$ and $V = \frac{4}{3}\pi r^3$.]

Solution

(i) Let A be the surface area of the balloon in m^2.
Let r be the radius of the balloon in m.

$\frac{dA}{dt} = 2$

$A = 4\pi r^2$

When $r = 3$,

$\frac{dA}{dr} = 8\pi r = 24\pi$

Comments

Define all variables that will be used, with their units.

Write down all given rates of change, and any relations between the variables.

Substitute $r = 3$ only in $\frac{dA}{dx}$, and not in $A = 4\pi r^2$, since r is a variable which takes a specific value of 3 in the question, not a constant.

Alternatively, find the expression for $\frac{dr}{dt}$ first, $\frac{dr}{dt} = \frac{1}{4\pi r}$, then substitute $r = 3$.

$$\frac{dr}{dt}=\frac{dr}{dA}\times\frac{dA}{dt}$$
$$=\left(1\div\frac{dA}{dr}\right)\times\frac{dA}{dt}$$
$$=\frac{1}{24\pi}\times 2$$
$$=\frac{1}{12\pi}$$

Identify what you need to find and what is given: $\frac{dr}{dt}$ given $\frac{dA}{dt}$ and $\frac{dA}{dr}$. Connect them using the Chain Rule. You can think of the dA as being "cancelled out".

Hence, the radius is increasing at $\frac{1}{12\pi}$ m s^{-1}. ■

Remember to include units in your final answer. Do not write units in $\frac{dr}{dt}$, since r and t are already in terms of m and s respectively.

(ii) Let V be the volume of the balloon in m^3.

$V=\frac{4}{3}\pi r^3$

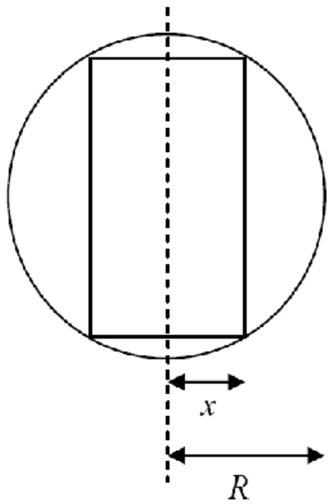
When $r=3$,

$\frac{dV}{dr}=4\pi r^2=36\pi$

$\frac{dV}{dt}=\frac{dV}{dr}\times\frac{dr}{dt}=36\pi\times 2=72\pi$

Hence, the volume is increasing at 72π m^3 s^{-1}. ■

Example 5 [SRJC08/Prelim/I/6]



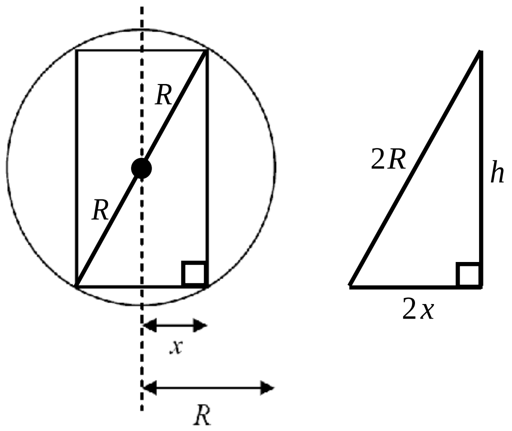
The diagram below shows the cross-section of a cylinder of radius x that is inscribed in a sphere of fixed internal radius R.

Show that $A^2=16\pi^2x^2\left(R^2-x^2\right)$, where A is the curved surface area of the cylinder.

Prove that, as x varies, the maximum value of A is obtained when the height of the cylinder is equal to its diameter.

Solution

Comments



Draw a line across the rectangular cross-section of the cylinder. Introducing right-angled triangles is useful because it allows the use of **Pythagoras' Theorem** and **trigonometric ratios**.

Let h be the height of the cylinder.

Define all variables that will be used, and are not already defined in the question.

By Pythagoras' Theorem,

$$(2R)^2 = h^2 + (2x)^2$$

$$h = 2\sqrt{R^2 - x^2}$$

The curved surface area of a cylinder is its circumference × height $= 2\pi rh = 2\pi xh$.

Since the expression to be proven is in terms of R and x only, express h in terms of R and x first.

$$\begin{aligned} A &= 2\pi xh \\ &= 2\pi x\left(2\sqrt{R^2 - x^2}\right) \\ &= 4\pi x\sqrt{R^2 - x^2} \end{aligned}$$

$A^2 = 16\pi^2 x^2 \left(R^2 - x^2\right)$ ■ (shown)

$$\begin{aligned} \frac{dA}{dx} &= 4\pi \frac{d}{dx} x\sqrt{R^2 - x^2} \\ &= 4\pi\left[\sqrt{R^2 - x^2} + x\left(\frac{-2x}{2\sqrt{R^2 - x^2}}\right)\right] \\ &= 4\pi\left(\sqrt{R^2 - x^2} - \frac{x^2}{\sqrt{R^2 - x^2}}\right) \end{aligned}$$

Use the Product Rule. The question states A is to be maximized as x varies, so find the appropriate derivative $\frac{dA}{dx}$.

Since R is a constant, $\frac{d}{dx}R^2 = 0$.

$= \frac{4\pi}{\sqrt{R^2 - x^2}}\left(R^2 - 2x^2\right)$

When $\frac{dA}{dx} = 0$,

$R^2 - 2x^2 = 0$

Factorize $\frac{dA}{dx}$ by removing terms which you know are non-zero, such as constants and $\left[f(x)\right]^2$, $\sqrt{f(x)}$, where $f(x) \neq 0$ for all x. e.g. $x \neq R$ here ($\because x = R \Rightarrow h = 0$), so $\sqrt{R^2 - x^2} > 0$.

This allows cancelling out of terms when $\frac{dA}{dx}$ is equated to 0.

$x^2 = \frac{R^2}{2}$

$x = \frac{R}{\sqrt{2}}$ or $-\frac{R}{\sqrt{2}}$ (rejected $\because x > 0$)

$x = \frac{R}{\sqrt{2}}$

To prove A is maximum when $h = 2x$, first let $\frac{dA}{dx} = 0$, then solve for x.

x	$\left(\frac{R}{\sqrt{2}}\right)^-$	$\frac{R}{\sqrt{2}}$	$\left(\frac{R}{\sqrt{2}}\right)^+$
$\frac{dA}{dx}$	> 0	$= 0$	< 0

Hence, $x = \frac{R}{\sqrt{2}}$ maximizes A.

Always verify the nature of the stationary point. Here, $\frac{dA}{dx}$ is difficult to differentiate, so use the First Derivative Test instead.

Since there is only 1 value of x which gives a stationary point, it must be the "maximum value" to be proven.

To carry out the test, choose some $R > 0$, e.g. $R = 1$.

When $x = \frac{R}{\sqrt{2}}$, $R = \sqrt{2}x$,

$h = 2\sqrt{R^2 - x^2}$

$= 2\sqrt{\left(\sqrt{2}x\right)^2 - x^2}$

$= 2\sqrt{x^2}$

$= 2x$ or $-2x$ (rejected $\because x > 0$ and $h > 0$)

$h = 2x =$ diameter of cylinder ■ (proven)

Express R in terms of x, then substitute for R in the equation $h = 2\sqrt{R^2 - x^2}$.

Integration

Basic Formulae

Let u and v be functions of x.

$$\int (u+v)\,dx = \int u\,dx + \int v\,dx$$

$$\int \frac{d}{dx} f(x)\,dx = f(x) + c$$ Integration reverses differentiation.

$$\frac{d}{dx}\int f(x)\,dx = f(x)$$ Differentiation reverses integration.

If $f(x)$ is a function which is defined and continuous for all $a \le x \le b$ and

$$\int f(x)\,dx = F(x) + c$$

then the definite integral of $f(x)$ from a to b with respect to x is:

$$\int_a^b f(x)\,dx = \left[F(x)\right]_a^b = F(b) - F(a)$$

Integration of Standard Functions

Let n be an integer and a be a constant.
Unless otherwise stated, the derivatives below hold for all $x \in \mathbb{R}$.

Remember to include the **arbitrary constant** c for indefinite integrals.

NOT in MF15:

$f(x)$	$\int f(x)\,dx$	**Range of validity**
x^n	$\frac{x^{n+1}}{n+1}$	$n \neq -1$
$\frac{1}{x}$	$\ln\lvert x\rvert$	
e^x	e^x	
a^x	$\frac{a^x}{\ln a}$	

$f(x)$	$\int f(x)\,dx$
$\cos x$	$\sin x$
$\sin x$	$-\cos x$
$\sec^2 x$	$\tan x$

In MF15:

$f(x)$	$\int f(x)\,dx$	Range of validity
$\dfrac{1}{x^2+a^2}$	$\dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right)$	
$\dfrac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\left(\dfrac{x}{a}\right)$	$\lvert x\rvert < a$
$\dfrac{1}{x^2-a^2}$	$\dfrac{1}{2a}\ln\left(\dfrac{x-a}{x+a}\right)$	$x>a$
$\dfrac{1}{a^2-x^2}$	$\dfrac{1}{2a}\ln\left(\dfrac{a+x}{a-x}\right)$	$\lvert x\rvert < a$

$f(x)$	$\int f(x)\,dx$	Range of validity
$\tan x$	$\ln(\sec x)$	$\lvert x\rvert < \frac{1}{2}\pi$
$\cot x$	$\ln(\sin x)$	$0 < x < \pi$
$\operatorname{cosec} x$	$-\ln(\operatorname{cosec} x + \cot x)$	$0 < x < \pi$
$\tan x$	$\ln(\sec x + \tan x)$	$\lvert x\rvert < \frac{1}{2}\pi$

If $\int f(x)dx$ involves a **logarithm** ($\ln[g(x)]$, where $g(x)$ is a function of x), place a **modulus** around the expression within the logarithm, i.e. write $\ln|g(x)|$. This removes the need to specify the range of validity.

- e.g. $\int \cot x = \ln|\sin x| + c$

Techniques of Integration

- Let $f(x)$ be any function and $F(x)$ be its integral given in the table above. To find the integral of $f(g(x))\times g'(x)$, where $g(x)$ is a function of x, use the fact that by the **Chain Rule** from differentiation, $\dfrac{d}{dx}F(g(x)) = f(g(x))\times g'(x)$.

  Hence,

$$\int f(g(x))\times g'(x)\,dx = F(g(x)) + c$$

- **Trigonometric Formulae:** Used to integrate expressions with trigonometric functions.

NOT in MF15:	In MF15:
$\sin^2 x + \cos^2 x = 1$ $\tan^2 x + 1 = \sec^2 x$ $\cot^2 x + 1 = \operatorname{cosec}^2 x$	$\sin 2x = 2\sin x\cos x$ $\cos 2x = \cos^2 x - \sin^2 x$ $= 2\cos^2 x - 1$ $= 1 - 2\sin^2 x$ *(and others)*

- **Partial Fractions:** Used to integrate expressions with an algebraic fraction, which has a denominator that can be factorized (or is already factorized). Splitting the fraction into partial fractions can help integrate it, since the partial fractions will be of the form $\frac{f'(x)}{f(x)}$.

  Partial fraction decomposition formulae are given in MF15.

- **Integration by Parts:** Used to integrate expressions with a product of 2 distinct functions.

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$$

  Generally, choose u in the following order such that $\frac{dv}{dx}$ is easier to integrate:

  LIATE: Logarithmic, **I**nverse trigonometric, **A**lgebraic, **T**rigonometric, **E**xponential.

  - To find $\int f(x)dx$ when you can find $\int xf'(x)dx$, let $u = 1$ and $\frac{dv}{dx} = f(x)$.
  - Use algebraic manipulation if $\int f(x)dx$ becomes the only integral on both sides.

- **Integration by Substitution:** If a substitution is required, it will be given in the question. To find $\int f(x)dx$ with the substitution $x = g(u)$:

$$\int f(x) dx = \int f(g(u)) \frac{dx}{du} du$$

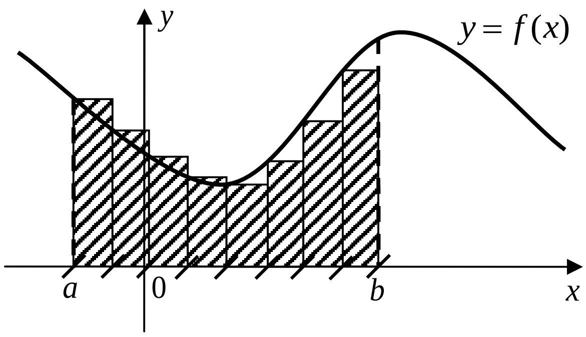
  For definite integrals,

$$\int_{x_1}^{x_2} f(x)dx = \int_{u_1}^{u_2} f(g(u)) \frac{dx}{du} du$$

  where $x_1 = g(u_1)$ and $x_2 = g(u_2)$.

  - For indefinite integrals, express your final answer in terms of x, not u.

Applications of Integration

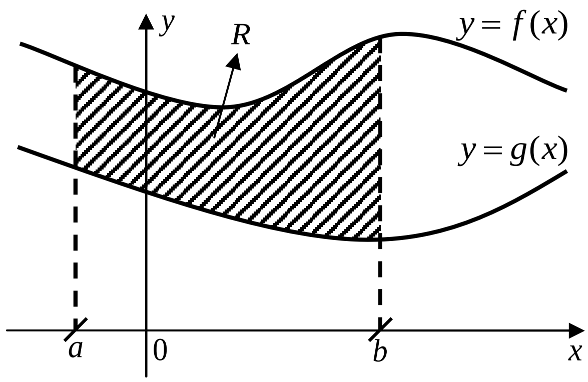


If $f(x) \geq 0$ for all $a \leq x \leq b$, and R is the area between the curve and the x-axis from $x = a$ to $x = b$,

The area of R can be approximated by the sum of areas of n rectangles, as shown.

The **exact** area of R is the limit of this sum of areas as $n \to \infty$, i.e. the definite integral:

$$\int_a^b f(x)\,dx$$



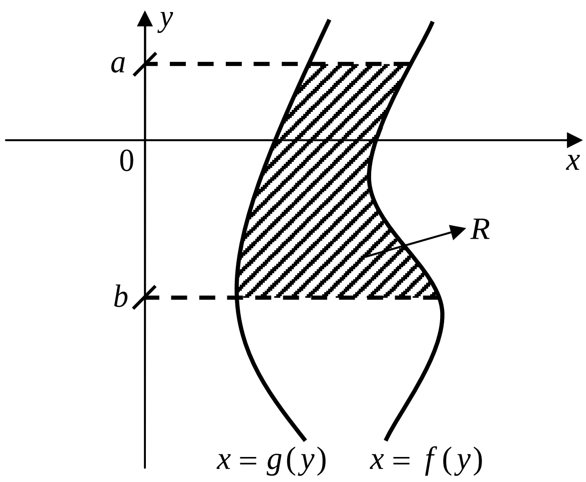
If $f(x) \geq g(x) \geq 0$ for all $a \leq x \leq b$,

Area of R:

$$\int_a^b \left[f(x) - g(x) \right] dx$$

Volume of solid formed when R is revolved completely about the x-axis:

$$\pi \int_a^b \left[\left(f(x) \right)^2 - \left(g(x) \right)^2 \right] dx$$



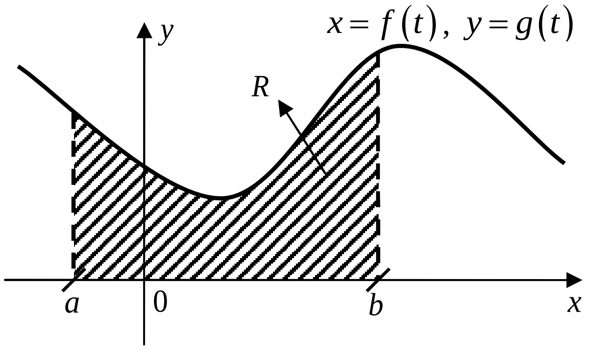
If $f(y) \geq g(y) \geq 0$ for all $a \leq x \leq b$,

Area of R:

$$\int_a^b \left[f(y) - g(y) \right] dx$$

Volume of solid formed when R is revolved completely about the y-axis:

$$\pi \int_a^b \left[\left(f(y) \right)^2 - \left(g(y) \right)^2 \right] dy$$

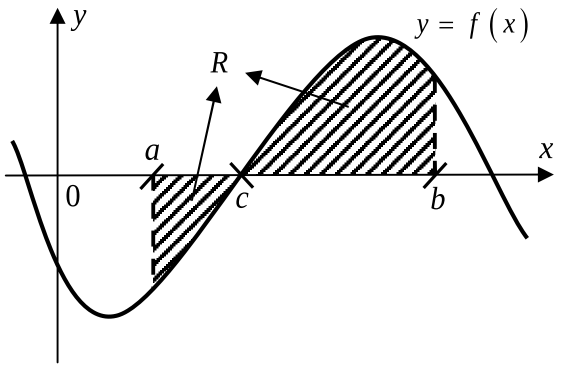


If a function is defined in **parametric** form as $x = f(t)$, $y = g(t)$, and $y = g(t) \geq 0$ for all $a \leq x \leq b$,

Area of R :

$$\int_p^q g(t)\frac{dx}{dt}\,dt$$

where $f(p) = a$ and $f(q) = b$.



If $y = f(x)$ has both positive and negative components within the limits of integration, split the area between the curve and the x-axis into its positive and negative parts.

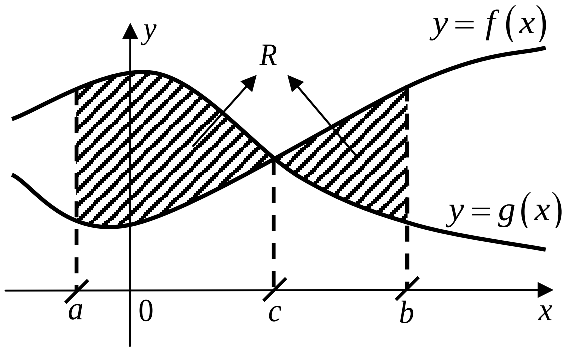
If $f(x) \leq 0$ for all $a \leq x \leq c$ and $f(x) \geq 0$ for all $c \leq x \leq b$,

Area of R :

$$\int_a^c -f(x)\,dx + \int_c^b f(x)\,dx$$

Or:

$$\int_a^b \left|f(x)\right| dx$$



If $y = f(x)$ and $y = g(x)$ intersect each other within the limits of integration, split the area between the 2 curves into where $f(x) > g(x)$ and where $g(x) > f(x)$.

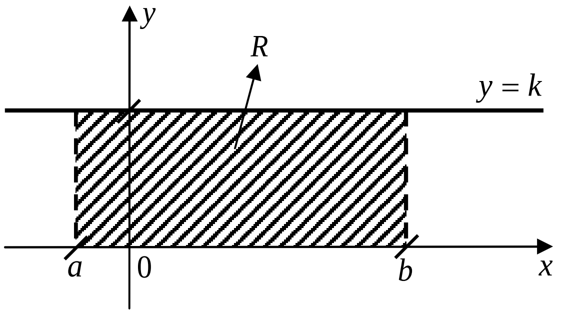
If $g(x) \geq f(x) \geq 0$ for all $a \leq x \leq c$ and $f(x) \geq g(x) \geq 0$ for all $c \leq x \leq b$,

Area of R :

$$\int_a^c \left[g(x) - f(x)\right] dx + \int_c^b \left[f(x) - g(x)\right] dx$$

Or:

$$\int_a^b \left|f(x) - g(x)\right| dx$$

	If $y = k$ for some $k \in \mathbb{R}$, $k \neq 0$, and $a < b$, Area of R : $k(b-a)$ The solid formed when R is revolved completely about the x-axis is a **cylinder** with radius k, height $b-a$, and volume: $\pi k^2 (b-a)$

- Use the GC to check your answer, or find an integral if an exact answer is not required.
  - [MATH] → [9], then key in fnInt(<expression>, X, <lower limit>, <upper limit>) e.g. to find $\int_0^1 x^2 dx$, key in fnInt$(X^2, X, 0, 1)$.
- Use the GC to sketch graphs of functions first before finding the areas or volumes required. For parametric functions, use [MODE] → PAR and key in T using [$X, T,$ n].

<u>Example 1</u> [N08/I/5]

(i) Find the exact value of $\int_0^{\frac{1}{\sqrt{3}}} \frac{1}{1+9x^2} dx$.

(ii) Find, in terms of n and e, $\int_1^e x^n \ln x dx$, where $n \neq -1$.

Solution

(i) $\int_0^{\frac{1}{\sqrt{3}}} \frac{1}{1+9x^2} dx = \frac{1}{9}\int_0^{\frac{1}{\sqrt{3}}} \frac{1}{x^2 + \left(\frac{1}{3}\right)^2} dx$

$= \frac{1}{9}\left[3\tan^{-1}(3x)\right]_0^{\frac{1}{\sqrt{3}}}$

$= \frac{1}{3}\left(\tan^{-1}\sqrt{3} - \tan^{-1} 0\right)$

$= \frac{1}{3}\left(\frac{\pi}{3}\right)$

$= \frac{\pi}{9}$ ■

Comments

Recognize when the integral is of a form given in MF15, here $\frac{1}{x^2+a^2}$. Either extract $\frac{1}{9}$ to get the x^2 term, as shown, or extract $\frac{1}{3}$ (since $\frac{d}{dx}(3x) = 3$) to get:

$$\frac{1}{3}\int_0^{\frac{1}{\sqrt{3}}} \frac{3}{1^2 + (3x)^2} dx$$

Then, apply the formula in MF15.

Check your answer using GC.

(ii) Let $u = \ln x$, $\dfrac{dv}{dx} = x^n$.

Then $\dfrac{du}{dx} = \dfrac{1}{x}$, $v = \dfrac{x^{n+1}}{n+1}$.

$$\int_1^e x^n \ln x dx = \left[\frac{x^{n+1}\ln x}{n+1}\right]_1^e - \int_1^e \frac{x^n}{n+1} dx$$

$$= \left[\frac{x^{n+1}\ln x}{n+1}\right]_1^e - \frac{1}{n+1}\int_1^e x^n dx$$

$$= \left[\frac{x^{n+1}\ln x}{n+1} - \frac{x^{n+1}}{(n+1)^2}\right]_1^e$$

$$= \left[\frac{e^{n+1}}{n+1} - \frac{e^{n+1}}{(n+1)^2}\right] - \left[0 - \frac{1}{(n+1)^2}\right]$$

$$= \frac{e^{n+1}}{n+1} - \frac{e^{n+1}}{(n+1)^2} + \frac{1}{(n+1)^2}$$

$$= \frac{e^{n+1}(n+1) - e^{n+1} + 1}{(n+1)^2}$$

$$= \frac{ne^{n+1}+1}{(n+1)^2} \blacksquare$$

Here, $\ln x$ is **L**ogarithmic and x^n is **A**lgebraic, so by LIATE, choose $u = \ln x$. Write out u, $\dfrac{dv}{dx}$, $\dfrac{du}{dx}$, and v.

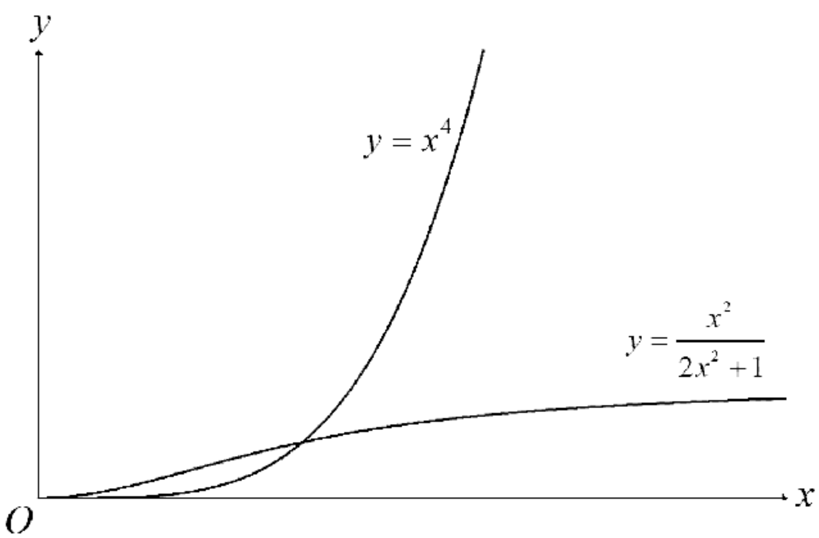
n is a constant, so treat it as a constant when integrating.

For definite integrals, remember to keep the limits on every term (integrals and results of integrals).

Check your answer using GC by choosing some n, $n \neq -1$, e.g. $n = 2$.

<u>Example 2</u> [HCI08/Prelim/I/10]

The diagram below shows the graphs of $y = \dfrac{x^2}{2x^2+1}$ and $y = x^4$, for $x \geq 0$. R is the region bounded by the y-axis, both curves and $y = 1$.



(i) By substituting $y=\frac{1}{2}\sin^2\theta$, find the exact value of

$$\int_0^{\frac{1}{4}}\sqrt{\frac{y}{1-2y}}dy$$

(ii) Verify that the two curves intersect at the point $\left(\frac{1}{\sqrt{2}},\frac{1}{4}\right)$. Hence or otherwise find the exact area of region R.

(iii) Find the volume of the solid generated when R is rotated through 2π radians about the y-axis, giving your answer correct to 3 decimal places.

Solution

(i)
$$\frac{dy}{d\theta}=\frac{1}{2}(2\sin\theta)(\cos\theta)$$
$$=\sin\theta\cos\theta$$

When $y=0$, $\sin\theta=0\Rightarrow\theta=0$

When $y=\frac{1}{4}$, $\sin\theta=\frac{\sqrt{2}}{2}$ or $-\frac{\sqrt{2}}{2}$ (rejected) $\Rightarrow\theta=\frac{\pi}{4}$

$$\int_0^{\frac{1}{4}}\sqrt{\frac{y}{1-2y}}dy=\int_0^{\frac{\pi}{4}}\sqrt{\frac{\frac{1}{2}\sin^2\theta}{1-\sin^2\theta}}\left(\frac{dy}{d\theta}\right)d\theta$$

$$=\int_0^{\frac{\pi}{4}}\sqrt{\frac{\frac{1}{2}\sin^2\theta}{\cos^2\theta}}\sin\theta\cos\theta d\theta$$

$$=\frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{4}}\frac{\sin\theta}{\cos\theta}\sin\theta\cos\theta d\theta$$

$$=\frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{4}}\sin^2\theta d\theta$$

Comments

To find $\int f(y)dx$ with the substitution $y=g(\theta)$, recall the general formula:

$$\int_{y_1}^{y_2}f(y)dy=\int_{\theta_1}^{\theta_2}f(\theta)\frac{dy}{d\theta}d\theta$$

where $y_1=g(\theta_1)$ and $y_2=g(\theta_2)$.

For a definite integral, change 4 things from the original integral:

1. Limits of integration.
2. The function to be integrated.
3. Multiply by $\frac{dy}{d\theta}$.
4. Integrate with respect to θ, not y.

Find 1. and 3. first in your working.

Write out the original integral, and make all 4 changes above at the same time. Simplify the substituted integral.

$$= \frac{1}{2\sqrt{2}} \int_0^{\frac{\pi}{4}} \left(1 - \cos 2\theta\right) d\theta$$

$$= \frac{1}{2\sqrt{2}} \left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{4}}$$

$$= \frac{\sqrt{2}}{4}\left[\left(\frac{\pi}{4} - \frac{1}{2}\right) - \left(0 - 0\right)\right]$$

$$= \frac{\sqrt{2}(\pi - 2)}{16} \text{ units}^2 \ \blacksquare$$

To integrate $\sin^2 x$ or $\cos^2 x$, use the double-angle formula for $\cos 2x$ to convert them.

$$\cos 2x = 1 - 2\sin^2 x = 2\cos^2 x - 1$$

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$\cos^2 x = \frac{1}{2}\left(1 + \cos 2x\right)$$

(ii) When $x = \frac{1}{\sqrt{2}}$,

$$x^4 = \left(\frac{1}{\sqrt{2}}\right)^4 = \frac{1}{4}$$

$$\frac{x^2}{2x^2 + 1} = \frac{\frac{1}{2}}{2\left(\frac{1}{2}\right) + 1} = \frac{1}{4}$$

Hence, the curves intersect at the point $\left(\frac{1}{\sqrt{2}}, \frac{1}{4}\right)$. $\blacksquare$ (verified)

The question only requires you to verify (NOT show) that at $\left(\frac{1}{\sqrt{2}}, \frac{1}{4}\right)$, the curves intersect. Hence, just show that the point lies on both curves.

If the question requires us to find all intersection points, you must use the equation $x^4 = \frac{x^2}{2x^2 + 1}$ and solve for x.

$$y = x^4 \Rightarrow x = y^{\frac{1}{4}}$$

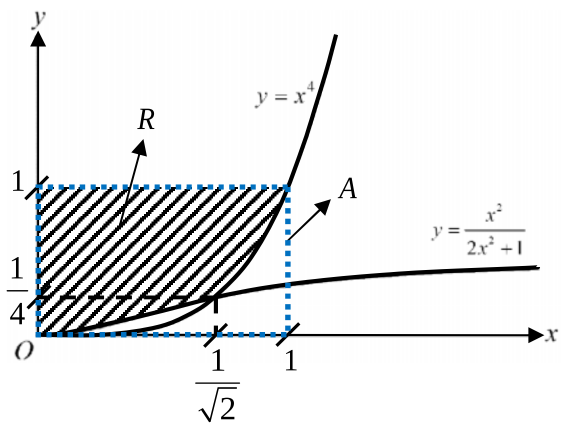
$$y = \frac{x^2}{2x^2 + 1}$$

$$\Rightarrow 2yx^2 + y - x^2 = 0$$

$$\Rightarrow \left(2y - 1\right)x^2 = -y$$

$$\Rightarrow x = \sqrt{\frac{y}{1 - 2y}}$$

or $-\sqrt{\frac{y}{1 - 2y}}$ (rejected $\because x > 0$)



Area of $R=\int_{0}^{\frac{1}{4}}\sqrt{\dfrac{y}{1-2y}}\,dy+\int_{\frac{1}{4}}^{1}y^{\frac{1}{4}}\,dy$

Look at the graph above to decide what to integrate. From $y=0$ to $y=\dfrac{1}{4}$, R is bounded by the y-axis and $x=\sqrt{\dfrac{y}{1-2y}}$.

From $y=\dfrac{1}{4}$ to $y=1$, R is bounded by the y-axis and $x=y^{\frac{1}{4}}$. Hence, split the integral into these 2 regions.

The method shown here integrates with respect to y, For each curve, express x in terms of y, then integrate with the limits from the y-axis.

$$=\frac{\sqrt{2}(\pi-2)}{16}+\frac{4}{5}\left[y^{\frac{5}{4}}\right]_{\frac{1}{4}}^{1}$$

$$=\frac{\sqrt{2}(\pi-2)}{16}+\frac{4}{5}-\frac{4}{5}\left(\frac{1}{4}\right)^{\frac{5}{4}}$$

$$=\frac{\sqrt{2}(\pi-2)}{16}+\frac{4}{5}-\frac{4}{5}\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)^{\frac{1}{4}}$$

$$=\frac{\sqrt{2}(\pi-2)}{16}+\frac{4}{5}-\frac{1}{5}\left(\frac{1}{4^{\frac{1}{4}}}\right)$$

$$=\frac{\sqrt{2}(\pi-2)}{16}+\frac{4}{5}-\frac{1}{5\sqrt{2}}$$

$$=\frac{4}{5}+\frac{\pi\sqrt{2}}{16}-\frac{\sqrt{2}}{8}-\frac{\sqrt{2}}{10}$$

$$=\left(\frac{4}{5}+\frac{\pi\sqrt{2}}{16}-\frac{9\sqrt{2}}{40}\right)\text{ units}^2 \ \blacksquare$$

We know the value of the first integral, $\int_{0}^{\frac{1}{4}}\sqrt{\dfrac{y}{1-2y}}\,dy$, from part (i).

Alternative method

Let A be the region bounded by the axes, $x=1$ and $y=1$, as shown by the dotted rectangle in the graph above. A is a rectangle with area 1.

Area of region R
= Area of region A
− Area of region outside R and inside A

$$=1-\left[\int_{0}^{\frac{1}{\sqrt{2}}}\frac{x^2}{2x^2+1}\,dx+\int_{\frac{1}{\sqrt{2}}}^{1}x^4\,dx\right]$$

Use long division on $\dfrac{x^2}{2x^2+1}$ to get an integral of the form $\dfrac{1}{a^2+x^2}$, then use MF15.

The alternative method integrates with respect to x. Each curve is already expressed as y in terms of x, hence integrate with the limits from the x-axis.

(iii) Volume required

$$=\pi\left[\int_0^{\frac{1}{4}}\frac{y}{1-2y}\,dy+\int_{\frac{1}{4}}^{1}y^{\frac{1}{2}}\,dy\right]$$

$=1.984$ units^3 (3 d.p.) ■

Volume of revolution about y-axis is given by $\pi\int x^2 dy$. Again, split the integral into 2 regions, then express x^2 in terms of y for each region.

The question states “3 decimal places” but not “exact”, suggesting that you find the volume directly using GC.

Differential Equations

Below, **DE** denotes **Differential Equation**.

Forms of DEs and General Solutions

Convert any given DE into one of these 3 forms.

<table>
<tr><th>Form</th><th colspan="2">General Solution</th></tr>
<tr><td>$\frac{dy}{dx}=f(x)$</td><td>Integrate both sides with respect to x.</td><td>$y=\int f(x)\,dx$</td></tr>
<tr><td>$\frac{dy}{dx}=f(y)$</td><td>Divide both sides by $f(y)$.

Integrate both sides with respect to x.

You can imagine the dx's cancelling out:
$$\int\left(\frac{1}{f(y)}\right)\frac{dy}{dx}\,dx=\int\frac{1}{f(y)}\,dy$$</td><td>$\left(\frac{1}{f(y)}\right)\frac{dy}{dx}=1$

$\int\frac{1}{f(y)}\,dy=x$</td></tr>
<tr><td>$\frac{d^2y}{dx^2}=f(x)$</td><td>Integrate both sides with respect to x.

Say $\int f(x)dx=g(x)+A$,
where A is an arbitrary constant.

Integrate both sides again with respect to x.

Say $\int g(x)dx=h(x)+B$,
where B is an arbitrary constant.

There are 2 arbitrary constants in the general solution of this form.</td><td>$\frac{dy}{dx}=\int f(x)dx$

$\frac{dy}{dx}=g(x)+A$

$y=\int\left(g(x)+A\right)dx$
$=\int g(x)dx+Ax$

$y=h(x)+B+Ax$

$y=h(x)+Ax+B$</td></tr>
</table>

Particular Solutions

- To find a "particular solution" to a DE, given some initial conditions,
    1. Substitute the initial conditions into the general solution.
    2. Find the value of the arbitrary constants.
    3. Rewrite the general solution with the known values for the arbitrary constants.

- To draw a **family of solution curves**,
  1. Substitute at least 3 distinct values for the arbitrary constant c.
     Generally, ensure you include these 3 cases: $c=0$, $c<0$, and $c>0$.
  2. Draw the graphs using GC, indicating intercepts, asymptotes, turning points, etc.

Solving DEs by Substitution

If a DE is given as $\frac{dy}{dx}=f(x,y)$ and a substitution $v=g(x,y)$ is given to solve this DE, where $f(x,y)$ and $g(x,y)$ are functions of x and y, aim to express the DE in terms of v and x only (without y).

1. Differentiate the given substitution with respect to x to get $\frac{dy}{dx}$ in terms of $\frac{dv}{dx}$, v and x.
2. In the original DE, express y and $\frac{dy}{dx}$ in terms of $\frac{dv}{dx}$, v and x.
3. Convert the substituted DE into the form required or one of the 3 standard forms above.
4. Solve the DE.
5. Express v in terms of x and y, such that your final answer is in terms of y and x only.

Modelling

- In modelling questions, focus on forming the DE correctly as a basis for your answer.
  1. Define all variables that will be used, and are not already defined in the question.
  2. Identify the **rate of change**, say $\frac{dx}{dt}$. Then the L.H.S. of the DE will be $\frac{dx}{dt}$.
  3. Rate of change = **Rate of increase** − **Rate of decrease**,
     where Rate of increase and Rate of decrease are both **positive**.

- If $\frac{dx}{dt}$ is **proportional** to something, e.g. x^2t, then $\frac{dx}{dt}\propto x^2t$. Write:
  $\frac{dx}{dt}=kx^2t$, where k is a positive constant.

- If $\frac{dx}{dt}$ is **inversely proportional** to something, e.g. x^2t, then $\frac{dx}{dt}\propto\frac{1}{x^2t}$. Write:
  $\frac{dx}{dt}=\frac{k}{x^2t}$, where k is a positive constant.

- Look out for key phrases in the question such as "water flows out at a rate of …", which indicates a rate of decrease, or "is proportional to", which indicates proportionality.

<u>Example 1</u> [N08/I/4]

(i) Find the general solution of the differential equation

$$\frac{dy}{dx}=\frac{3x}{x^2+1}$$

(ii) Find the particular solution of the differential equation for which $y=2$ when $x=0$.

(iii) What can you say about the gradient of every solution curve as $x\to\pm\infty$?

(iv) Sketch, on a single diagram, the graph of the solution found in part (ii), together with 2 other members of the family of solution curves.

Solution

(i) $\frac{dy}{dx}=\frac{3x}{x^2+1}=\frac{3}{2}\left(\frac{2x}{x^2+1}\right)$

$y=\frac{3}{2}\int\frac{2x}{x^2+1}dx$

$=\frac{3}{2}\ln(x^2+1)+c,$

where c is an arbitrary constant. ■

(ii) When $y=2$, $x=0$,

$c=y-\frac{3}{2}\ln(x^2+1)$

$=2-\frac{3}{2}\ln 1$

$=2$

$\therefore y=\frac{3}{2}\ln(x^2+1)+2$ ■

(iii) As $x\to\pm\infty$, the gradient of every solution curve tends to 0. ■

Comments

The DE is of the form $\frac{dy}{dx}=f(x)$, so just integrate both sides. Remember the arbitrary constant c.

Write this line for proper presentation.

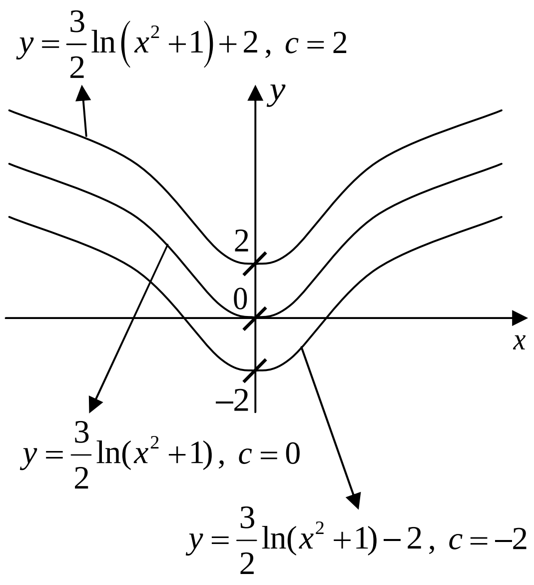
1. Substitute the initial conditions into the general solution.
2. Find the value of the arbitrary constants.
3. Rewrite the general solution with the known values for the arbitrary constants.

The gradient of every solution curve is $\frac{dy}{dx}=\frac{3x}{x^2+1}$, from the DE. As $x\to\pm\infty$, because the denominator x^2+1 will "grow faster" than the numerator $3x$, $\frac{dy}{dx}\to 0$.

OR

Use the GC to plot the graph of $y=\frac{3x}{x^2+1}$. Observe the asymptote $y=0$ as $x\to\pm\infty$.

(iv)



Choose 2 other solution curves which are representative of all values of c. Since $c=2$ is a required curve, choose $c=0$ and some $c<0$, here $c=-2$.

Use the GC to draw the graphs of the solution curves.

When the general solution is of the form $y=f\left(x\right)+c$, the solution curves are translations of each other along the y-axis.

The **gradient is 0 at $x=0$**, since $\frac{dy}{dx}=\frac{3x}{x^2+1}=0$. At some zoom levels on the GC, e.g. ZStandard, the curve looks like a “V” with sharp tangents at $x=0$.

The graph has **no horizontal asymptote**, even though from part (iii), the gradient of every solution curve tends to 0 as $x\to\pm\infty$.

<u>Example 2</u> [NYJC08/Prelim/II/1 (Part)]

Show that the differential equation

$$x^2\frac{dy}{dx}-2xy+3=0$$

may be reduced by means of the substitution $y=ux^2$ to

$$\frac{du}{dx}=-\frac{3}{x^4}$$

Hence, or otherwise, find the general solution for y in terms of x.

Solution

$y=ux^2$

$$\frac{dy}{dx}=u\left(2x\right)+\frac{du}{dx}\left(x^2\right)$$

$$=2ux+x^2\frac{du}{dx}$$

Comments

1. Differentiate the given substitution with respect to x to get $\frac{dy}{dx}$ in terms of $\frac{du}{dx}$, u and x.

$x^2\left(2ux + x^2\dfrac{du}{dx}\right) - 2x\left(ux^2\right) + 3 = 0$

$x^4\dfrac{du}{dx} + 2ux^3 - 2ux^3 = -3$

$\dfrac{du}{dx} = -\dfrac{3}{x^4}$ ■ (shown)

2. In the original DE, express y and $\dfrac{dy}{dx}$ in terms of $\dfrac{du}{dx}$, u and x.
3. Convert the substituted DE into the form required.

$\dfrac{du}{dx} = -\dfrac{3}{x^4}$

4. Solve the DE.

$u = \int -\dfrac{3}{x^4}\,dx = -3\int x^{-4}\,dx$

$= -3\left(\dfrac{x^{-3}}{-3}\right) + c,$

The DE is of the form $\dfrac{dy}{dx} = f(x)$, so just integrate both sides. Remember the arbitrary constant c.

where c is an arbitrary constant.

Write this line for proper presentation.

$u = \dfrac{1}{x^3} + c$

$y = ux^2 \Rightarrow u = \dfrac{y}{x^2}$

$\dfrac{y}{x^2} = \dfrac{1}{x^3} + c$

$y = \dfrac{1}{x} + cx^2$ ■

Express u in terms of x and y, such that your final answer is in terms of y and x only.

Example 3 [HCI08/Prelim/I/7]

A hospital patient is receiving a certain drug through a drip at a constant rate of 50 mg per hour. The rate of loss of the drug from the patient's body is proportional to x, where x (in mg) is the amount of drug in the patient at time t (in hours). Form a differential equation connecting x and t and show that

$$x = \frac{1}{k}\left(50 - Ae^{-kt}\right),$$

where A and k are constants.

Given that $k = \dfrac{1}{20}$,

(i) Find the time needed for the amount of drug in the patient to reach 200 mg, if initially there is no trace of this drug in the patient's body.

(ii) When there is 80 mg of the drug in the patient's body, the drip is disconnected. Assuming that the rate of loss remains the same, find the time taken for the amount of drug in the patient to fall from 80 mg to 20 mg.

Solution

$\frac{dx}{dt} = 50 - kx,$
where k is a positive constant. ■

Comments

Rate of increase is constant at 50. Rate of decrease is proportional to x, so write it as kx, where k is a positive constant.

Now, Rate of change
= Rate of increase − Rate of decrease

$\frac{1}{50-kx}\left(\frac{dx}{dt}\right) = 1$

$\int \frac{1}{50-kx}\,dx = \int 1\,dt$

$-\frac{1}{k}\int \frac{-k}{50-kx}\,dx = \int 1\,dt$

The DE is of the form $\frac{dy}{dx} = f\left(y\right)$, so divide both sides by $f(y)$, then integrate both sides. Remember the arbitrary constant c.

$-\frac{1}{k}\ln\left|50-kx\right| = t + c,$
where c is an arbitrary constant.

$\ln\left|50-kx\right| = -kt - kc$

$\left|50-kx\right| = e^{-kt-kc}$

Evaluate both integrals and introduce the arbitrary constant in 1 step.

$50 - kx = \pm e^{-kc}\left(e^{-kt}\right) = Ae^{-kt},$
where $A = \pm e^{-kc}$ is an arbitrary constant.

$kx = 50 - Ae^{-kt}$

$x = \frac{1}{k}\left(50 - Ae^{-kt}\right)$ ■ (shown)

The modulus is eliminated by transferring it to the arbitrary constant $A = \pm e^{-kc}$.

(i) $x = 20\left(50 - Ae^{-\frac{t}{20}}\right)$

When $t = 0$, $x = 0$,
$0 = 20\left(50 - Ae^{0}\right)$

Use the information given: $k = \frac{1}{20}$. Interpret the question: "Initially" means $t = 0$, "no trace" means $x = 0$.

1. Substitute the initial conditions into the general solution.

$A = 50$

2. Find the value of the arbitrary constant.

$x = 20\left(50 - 50e^{-\frac{t}{20}}\right)$

3. Rewrite the general solution with the known value for the arbitrary constant.

When $x = 200$,

$$200 = 1000\left(1 - e^{-\frac{t}{20}}\right)$$

$$1 - e^{-\frac{t}{20}} = \frac{1}{5}$$

$$e^{-\frac{t}{20}} = \frac{4}{5}$$

$$-\frac{t}{20} = \ln\left(\frac{4}{5}\right)$$

$$t = -20\ln\left(\frac{4}{5}\right) = 4.46 \text{ hours (3 s.f.)} \blacksquare$$

Interpret the question: Find t when $x = 200$.

(ii)

$$\frac{dx}{dt} = -\frac{1}{20}x$$

$$-\frac{20}{x}\left(\frac{dx}{dt}\right) = 1$$

$$-20\int \frac{1}{x}\,dx = \int 1\,dt$$

"Rate of loss remains the same", but the drip is disconnected, hence the rate of increase is 0.

With the given value of k, rewrite the DE and solve it.

$$-20\ln|x| = t + B,$$

where B is an arbitrary constant.

Evaluate both integrals and introduce the arbitrary constant in 1 step.

Remember the arbitrary constant, but since c is already used, use another letter, say B.

When $t = 0$, $x = 80$,

$B = -20\ln 80$

$-20\ln|x| = t - 20\ln 80$

Interpret the question: Initially, there is 80 mg of drug, so $t = 0$ and $x = 80$. Find t when $x = 20$.

When $x = 20$,

$$\begin{aligned} t &= 20\ln 80 - 20\ln 20 \\ &= 20\ln 4 \\ &= 27.7 \text{ hours (3 s.f.)} \blacksquare \end{aligned}$$

Power Series

If a function $f(x)$ can be expanded as a power series:

$$\boldsymbol{f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots + \frac{x^n}{n!}f^{(n)}(0) + \ldots}$$ **(In MF15)**

where $f^{(n)}(0)$ is the nth derivative of $f(x)$, i.e. $\frac{d^n f(x)}{dx^n}$, evaluated at $x = 0$.

The expression on the RHS is known as the **Maclaurin's series,** a type of power series.

- In the formula above, x may be replaced with any function of x.
- The Maclaurin's series of the following functions are also in MF15:
    - $\boldsymbol{(1+x)^n}$ for $|x| < 1$ (This is the binomial expansion for rational index)
    - $\boldsymbol{e^x}$, $\boldsymbol{\sin x}$ and $\boldsymbol{\cos x}$ for all x
    - $\boldsymbol{\ln(1+x)}$ for $-1 < x \le 1$ (Note the range of valid x values)
- Maclaurin's series are **infinite** series.
- The $\boldsymbol{n}$ **th term** in the Maclaurin's series is

$$\frac{x^n}{n!}f^{(n)}(0)$$

FAQ

Q: Find the Maclaurin's series for $\boldsymbol{f(x)}$, up to the term in x^k.

A: 1. Differentiate the expression repeatedly. **Implicit differentiation** may be helpful (or necessary if the question states "By further differentiation of this result").

Find equations connecting all derivatives **up to the $\boldsymbol{k}$ th derivative** $\boldsymbol{\frac{d^k f(x)}{dx^k}}$.

2. Evaluate $f(x)$ and its derivatives at $\boldsymbol{x} = 0$ using the equations found. When $x = 0$,
    - $f(x) = f(0)$
    - $\frac{dy}{dx} = f'(0)$
    - $\frac{d^2y}{dx^2} = f''(0)$, etc. up to $\frac{d^k f(x)}{dx^k} = f^{(k)}(0)$.
3. Substitute $f(0)$, $f'(0)$, $f''(0)$ etc. into the formula for Maclaurin's series.

Write out each term in ascending powers of x, up to x^k.
4. Check your answer by substitute a small value of x, e.g. $x = 0.01$ into both the original expression and the Maclaurin's series.

Small Angle Approximations

Using the Maclaurin's series of $\sin x$, $\cos x$ and $\tan x$ up to the term in x^2,

$$\sin kx \approx kx$$
$$\cos kx \approx 1-\frac{(kx)^2}{2}$$
$$\tan kx \approx kx$$

where k is a constant. The approximation is accurate when $|x|$ is small, i.e. x is **close to 0**.

- To approximate e.g. $\sin\left(x+\frac{\pi}{4}\right)$ where x is small, do NOT write $\sin\left(x+\frac{\pi}{4}\right) \approx x+\frac{\pi}{4}$, since $x+\frac{\pi}{4}$ is not small. Instead, use the addition formula for $\sin$. Since x is small, the approximations for $\sin x$ and $\cos x$ are valid.

$$\begin{aligned}\sin\left(x+\frac{\pi}{4}\right) &= \sin x\cos\frac{\pi}{4}+\cos x\sin\frac{\pi}{4} \\ &\approx x\frac{\sqrt{2}}{2}+\left(1-\frac{x^2}{2}\right)\frac{\sqrt{2}}{2} \\ &= \frac{\sqrt{2}}{2}\left(1-x+\frac{x^2}{2}\right)\end{aligned}$$

Example 1 [HCI08/Promo/12]

Given that $y = x\tan^{-1}x$, prove that

$$\left(1+x^2\right)\frac{d^2y}{dx^2}+2x\frac{dy}{dx}-2y-2=0$$

By repeated differentiation, show that the first two non-zero terms of the Maclaurin's series for y are $x^2-\frac{1}{3}x^4$.

Hence evaluate $\lim_{x\to 0}\left(\frac{x-\tan^{-1}x}{x^3}\right)$.

Solution	**Comments**
$y = x\tan^{-1}x$ $\frac{dy}{dx} = \tan^{-1}x+\frac{x}{1+x^2}$	Differentiate y to get $\frac{dy}{dx}$.

$$\left(1+x^2\right)\frac{dy}{dx}=\left(1+x^2\right)\tan^{-1}x+x$$
$$\left(1+x^2\right)\frac{d^2y}{dx^2}+2x\frac{dy}{dx}=1+2x\tan^{-1}x+1$$
$$\left(1+x^2\right)\frac{d^2y}{dx^2}+2x\frac{dy}{dx}-2y-2=0 \blacksquare \text{ (proven)}$$

Since the expression to be proven contains $\left(1+x^2\right)\frac{d^2y}{dx^2}$, manipulate the equation to contain $\left(1+x^2\right)\frac{dy}{dx}$, then use **implicit differentiation**.

$$\left(1+x^2\right)\frac{d^3y}{dx^3}+2x\frac{d^2y}{dx^2}+2x\frac{d^2y}{dx^2}+2\frac{dy}{dx}-2\frac{dy}{dx}=0$$
$$\left(1+x^2\right)\frac{d^3y}{dx^3}+4x\frac{d^2y}{dx^2}=0$$

$$\left(1+x^2\right)\frac{d^4y}{dx^4}+2x\frac{d^3y}{dx^3}+4x\frac{d^3y}{dx^3}+4\frac{d^2y}{dx^2}=0$$
$$\left(1+x^2\right)\frac{d^4y}{dx^4}+6x\frac{d^3y}{dx^3}+4\frac{d^2y}{dx^2}=0$$

Use the Product Rule and Chain Rule, and remember:

$$\frac{d}{dx}(y)=\frac{dy}{dx}$$
$$\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d^2y}{dx^2}$$
$$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)=\frac{d^3y}{dx^3}\text{, etc.}$$

Since the question requires the series up to x^4, find equations connecting all derivatives up to $\frac{d^4y}{dx^4}$.

When $x=0$,

$$y=0, \frac{dy}{dx}=0, \frac{d^2y}{dx^2}=2, \frac{d^3y}{dx^3}=0, \frac{d^4y}{dx^4}=-$$
$$\therefore y=0+0x+2\left(\frac{x^2}{2!}\right)+0x^3-8\left(\frac{x^4}{4!}\right)+\dots$$
$$y=x^2-\frac{1}{3}x^4+\dots \blacksquare \text{ (shown)}$$

Evaluate y and its derivatives at $x=0$, using the equations found above.

Substitute the resulting values into the formula for Maclaurin's series.

$$\frac{x-\tan^{-1}x}{x^3}=\frac{x^2-x\tan^{-1}x}{x^4}$$
$$=\frac{x^2-\left(x^2-\frac{1}{3}x^4+\dots\right)}{x^4}$$
$$=\frac{\frac{1}{3}x^4+\dots}{x^4}$$

Convert the expression to contain $x\tan^{-1}x$. Substitute the series for $x\tan^{-1}x$ which has been found into the expression.

$$\lim_{x\to 0}\left(\frac{x-\tan^{-1}x}{x^3}\right)=\lim_{x\to 0}\left(\frac{\frac{1}{3}x^4+\dots}{x^4}\right)=\frac{1}{3} \blacksquare$$

As x tends to 0, the x^5, x^6 etc. terms get smaller faster than the $\frac{1}{3}x^4$ term, thus the $\frac{1}{3}x^4$ term dominates the numerator.

Example 2 [NYJC08/Prelim/II/3 (Part)]

A curve C is defined by the equation $2y\dfrac{dy}{dx} = y^2 - 1$ and $(0, 2)$ is a point on C.

(i) Show that at $x = 0$, $\dfrac{d^3y}{dx^3} = \dfrac{57}{256}$.
Find the Maclaurin's series of y up to and including the term in x^3.

(ii) Hence find the series expansion of e^y, up to and including the term in x^2.

Solution	Comments
(i) $2y\dfrac{dy}{dx} = y^2 - 1$ $2y\dfrac{d^2y}{dx^2} + 2\left(\dfrac{dy}{dx}\right)^2 = 2y\dfrac{dy}{dx}$ $y\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^2 = y\dfrac{dy}{dx}$	Differentiate the equation of C repeatedly. Since the question requires the series up to x^3, find equations connecting all derivatives up to $\dfrac{d^3y}{dx^3}$.
$y\dfrac{d^3y}{dx^3} + \left(\dfrac{dy}{dx}\right)\left(\dfrac{d^2y}{dx^2}\right) + 2\left(\dfrac{dy}{dx}\right)\left(\dfrac{d^2y}{dx^2}\right) = y\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^2$	Since differentiation here is with respect to x, by the Product Rule and Chain Rule: $\dfrac{d}{dx}\left(y\dfrac{dy}{dx}\right) = y\left(\dfrac{d^2y}{dx^2}\right) + \left(\dfrac{dy}{dx}\right)^2$
$y\dfrac{d^3y}{dx^3} + 3\left(\dfrac{dy}{dx}\right)\left(\dfrac{d^2y}{dx^2}\right) = y\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^2$	Evaluate y and its derivatives at $\boldsymbol{x} = 0$, using the equations found above.
When $x = 0$, $y = 2, \dfrac{dy}{dx} = \dfrac{3}{4}$ $4\dfrac{d^2y}{dx^2} + 2\left(\dfrac{3}{4}\right)^2 = 4\left(\dfrac{3}{4}\right) \Rightarrow \dfrac{d^2y}{dx^2} = \dfrac{15}{32}$ $2\dfrac{d^3y}{dx^3} + 3\left(\dfrac{3}{4}\right)\left(\dfrac{15}{32}\right) = 2\left(\dfrac{15}{32}\right) + \left(\dfrac{3}{4}\right)^2$ $\dfrac{d^3y}{dx^3} = \dfrac{57}{256}$ ■ (shown)	
$\therefore y = 2 + \dfrac{3}{4}x + \dfrac{15}{32}\left(\dfrac{x^2}{2!}\right) + \dfrac{57}{256}\left(\dfrac{x^3}{3!}\right) + \ldots$ $y = 2 + \dfrac{3}{4}x + \dfrac{15}{64}x^2 + \dfrac{19}{512}x^3 + \ldots$ ■	Substitute the resulting values into the formula for Maclaurin's series.

(ii)

$$e^y = e^{2+\frac{3}{4}x+\frac{15}{64}x^2+\dots} = e^2\left(e^{\frac{3}{4}x+\frac{15}{64}x^2+\dots}\right)$$

$$= e^2\left[1+\frac{3}{4}x+\frac{15}{64}x^2+\frac{1}{2!}\left(\frac{3}{4}x+\frac{15}{64}x^2\right)^2+\dots\right]$$

$$= e^2\left[1+\frac{3}{4}x+\frac{15}{64}x^2+\frac{1}{2}\left(\frac{3}{4}\right)^2x^2\right]+\dots$$

$$= e^2+\frac{3}{4}e^2x+\frac{15}{64}e^2x^2+\dots \blacksquare$$

Substitute the series for y which has been found into the expression. e^2 must be extracted. Otherwise,

$$e^y = 1+y+\frac{1}{2!}y^2+\frac{1}{3!}y^3+\dots$$

will have infinitely many constant terms, since each y contains a 2. Hence, this series cannot be simplified.

Example 3 [YJC08/Prelim/I/4iii]

It is given that $y=\tan\left(e^x-1\right)$.

(i) Prove that $\frac{d^2y}{dx^2}=\frac{dy}{dx}\left(1+2e^xy\right)$.

(ii) Find Maclaurin's series for y in ascending powers of x, up to and including the term in x^3.

(iii) Deduce the first three terms of the Maclaurin's series for $\frac{\sec^2\left(e^{-x}-1\right)}{e^x}$.

We will focus on part (iii) only.

From part (i): $\boldsymbol{\frac{dy}{dx}=e^x\sec^2\left(e^x-1\right)}$

From part (ii): $\boldsymbol{y=x+\frac{1}{2}x^2+\frac{1}{2}x^3+\dots}$

Solution

(iii)

$$\frac{dy}{dx}=e^x\sec^2\left(e^x-1\right)$$

$$=\frac{d}{dx}\left(x+\frac{1}{2}x^2+\frac{1}{2}x^3+\dots\right)$$

$$=1+x+\frac{3}{2}x^2+\dots$$

$$e^{-x}\sec^2\left(e^x-1\right)=\frac{\sec^2\left(e^x-1\right)}{e^x}$$

$$=1+\left(-x\right)+\frac{3}{2}\left(-x\right)^2+\dots$$

$$=1-x+\frac{3}{2}x^2+\dots \blacksquare$$

Comments

$\frac{dy}{dx}$ looks similar to $\frac{\sec^2(e^{-x}-1)}{e^x}$. To obtain the series of $\frac{dy}{dx}$, differentiate both sides of

$$y=x+\frac{1}{2}x^2+\frac{1}{2}x^3+\dots$$

Since $\frac{\sec^2(e^{-x}-1)}{e^x}$ has an e^{-x} term, substitute x with $-x$.

Questions on binomial expansion or power series commonly include differentiation and integration concepts.

Binomial Expansion

Binomial Expansion for Rational Index

$$\boldsymbol{(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{r!}x^r + \ldots}$$ **(In MF15)**

where n is either a non-integer rational number or a negative integer, and $|x| < 1$.

- The expansion is an **infinite** series.
- For the expansion to be valid, $|x| < 1$.
- Use the GC to check your answer by substituting some x within the range of validity. The actual value and the value given by the expansion should be approximately equal.
- Knowing "shortcut" expansions **(NOT in MF15)** can be convenient. For $|x| < 1$,

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots + x^r + \ldots$$
$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots + (-1)^r x^r + \ldots$$
$$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots + (r+1)x^r + \ldots$$

FAQ

Q1: Expand $(a+bx^c)^n$ in ascending powers of x up to the term in x^k, where a and b are constants and x^c is a positive power of x.

A1: 1. **Extract a^n** to convert the expression such that it **contains 1** as a term.

$$(a+bx^c)^n = a^n\left[1 + \frac{bx^c}{a}\right]^n$$

2. Use the binomial expansion for rational index, including only terms **up to x^k**.
Use the above "shortcut" expansions if possible to save time.
Write out **the entire expression** before simplifying to reduce careless mistakes.

3. Simplify the coefficients.

Q2: State the range of values of x for which the expansion of $\left(a+bx^{c}\right)^{n}$ is valid.

A2:
1. Look at the expression immediately before the binomial expansion was used. It should be of the form $\mathbf{[1+f(x)]^{n}}$.
2. For the expansion of $[1+f(x)]^{n}$, the range of validity is $|f(x)|<1$, i.e.

$$-1<f(x)<1$$

3. Simplify the inequality to give a range of values of x.

Q3: Expand an expression in ascending powers of x up to the term in x^{k}.

A3:
1. Express the expression as a **product** (or **sum**) of terms of the form $\left(a+bx^{c}\right)^{n}$, using partial fractions if required.
2. For each factor, follow the 3 steps in Answer (1) above.
3. **Multiply** (or **add**) the expansions for each term together.
4. Discard any terms which have a power of x greater than k.

Q4: State the range of values of x for which the expansion of an expression is valid.

A4:
1. Using Answer (2) above, find the range of values of x for which the expansion of **each term** of the form $\left(a+bx^{c}\right)^{n}$ is valid.
2. The set of values of x for which the entire expression is valid is then the **intersection** of the sets of values of x for each term.

Q5: Expand $\left(a+bx^{c}\right)^{n}$ in descending powers of x up to the term in x^{k}.

A5: Same as Answer (1) above, but for Step 1:

Extract $\left(bx^{c}\right)^{n}$ instead of a^{n} to convert each factor such that it **contains 1** as a term.

$$\left(a+bx^{c}\right)^{n}=\left(bx^{c}\right)^{n}\left[1+\frac{a}{bx^{c}}\right]^{n}$$

Q6: Find the r th term (or its coefficient) in the expansion of an expression.

A6: Use the fact that the $(r+1)^{th}$ term in the expansion is

$$\frac{n(n-1)\ldots(n-r+1)}{r!}x^{r}$$

Q7: By putting $x=a$, find an approximation for b, where a and b are constants.

A7:
1. Ensure a is within the range of values of x such that the expansion is valid.
2. Substitute $x=a$ into both the original expression and the expansion.
3. Manipulate the equation to obtain an approximation for b.

Example 1 [N2006/I/12 (Modified)]

(i) Express

$$f(x)=\frac{1+x-2x^2}{(2-x)(1+x^2)}$$

in partial fractions.

(ii) Expand $f(x)$ in ascending powers of x, up to and including the term in x^2.

(iii) Find the set of values of x for which the expansion is valid.

(iv) Deduce the equation of the tangent to the curve $y=f(x)$ at the point where $x=0$.

Solution	Comments
(i) *(working omitted)* $f(x)=\frac{1+x}{1+x^2}-\frac{1}{2-x}$ ■	Use the partial fractions decomposition for a non-repeated quadratic factor in the denominator (in MF15).
(ii) $f(x)=(1+x)(1+x^2)^{-1}-(2-x)^{-1}$	Express $f(x)$ as a sum of terms of the form $(a+bx^c)^n$.
$=(1+x)(1+x^2)^{-1}-2^{-1}\left(1-\frac{x}{2}\right)^{-1}$ $=(1+x)(1+x^2)^{-1}-\frac{1}{2}\left(1-\frac{x}{2}\right)^{-1}$	For each term, 1. **Extract** a^n to convert the expression such that it **contains 1** as a term.
$=(1+x)(1-x^2+\ldots)$ $-\frac{1}{2}\left[1+\frac{1}{2}x+\left(\frac{1}{2}x\right)^2+\ldots\right]$	2. Use the binomial expansion for rational index, including only terms **up to** x^2. Use the "shortcut" expansions of $(1-x)^{-1}$ and $(1+x)^{-1}$ to save time.
$=(1+x)(1-x^2+\ldots)$ $+\left(-\frac{1}{2}-\frac{1}{4}x-\frac{1}{8}x^2+\ldots\right)$	3. Simplify the coefficients.
$=1-x^2+x-\frac{1}{2}-\frac{1}{4}x-\frac{1}{8}x^2+\ldots$	Ignore $x(-x^2)=-x^3$ when simplifying, since the expansion should be up to x^2.
$=\frac{1}{2}+\frac{3}{4}x-\frac{9}{8}x^2+\ldots$ ■	**Write "…"** after the first few terms of an expansion (after any expansion in any step).

(iii) $\left|-\frac{1}{2}x\right|<1$ and $\left|x^2\right|<1$

$|x|<2$ and $|x|<1$

$-2<x<2$ and $-1<x<1$

For the expansion to be valid,
Set of values of $x=\{x\in\mathbb{R}:-1<x<1\}$ ■

Immediately before the binomial expansion was used, the expressions were $\left(1-\frac{1}{2}x\right)^{-1}$ and $\left(1+x^2\right)^{-1}$.

For the entire expansion to be valid, both binomial expansions involved must be valid, i.e. $\left|-\frac{1}{2}x\right|<1$ and $\left|x^2\right|<1$.

Find the intersection of the 2 sets of values, and give the final answer in set notation, since the question specified "set of values".

Use **set-builder notation**, NOT interval notation i.e. $(-1,1)$.

(iv) $y=\frac{1}{2}+\frac{3}{4}x-\frac{9}{8}x^2+\ldots$

$\frac{dy}{dx}=\frac{3}{4}-\frac{9}{4}x+\ldots$

When $x=0$, $y=\frac{1}{2}$ and $\frac{dy}{dx}=\frac{3}{4}$.

Equation of required tangent:

$y-\frac{1}{2}=\frac{3}{4}(x-0)$

$y=\frac{3}{4}x+\frac{1}{2}$ ■

To find the gradient of the tangent, find $\frac{dy}{dx}$ when $x=0$. To find the point of tangency, find y when $x=0$.

A straight line with gradient m and passing through a point with coordinates (x_1,y_1) has equation:

$$y-y_1=m(x-x_1)$$

Questions on binomial expansion or power series commonly include differentiation and integration concepts.

Example 2 [HCI08/Prelim/I/6]

Expand $\left(\frac{1-x}{1+x}\right)^n$ in ascending powers of x up to and including the term in x^2.

State the set of values of x for which the series expansion is valid.

Hence find an approximation to the fourth root of $\frac{19}{21}$, in the form $\frac{p}{q}$, where p and q are positive integers.

Solution

$$\left(\frac{1-x}{1+x}\right)^n = (1-x)^n (1+x)^{-n}$$

$$= \left[1 + n(-x) + \frac{n(n-1)}{2!}(-x)^2 + \dots\right] \times \left[1 + (-n)x + \frac{(-n)(-n-1)}{2!}x^2 + \dots\right]$$

$$= \left[1 - nx + \frac{n(n-1)}{2}x^2 + \dots\right] \times \left[1 - nx + \frac{n(n+1)}{2}x^2 + \dots\right]$$

$$= 1 - nx + \frac{n(n+1)}{2}x^2 - nx + n^2x^2 + \frac{n(n-1)}{2}x^2 + \dots$$

$$= 1 - 2nx + 2n^2x^2 + \dots \blacksquare$$

For the expansion to be valid, $|x| < 1$.

Set of values of $x = \{x \in \mathbb{R} : -1 < x < 1\}$ ■

Comments

Express $f(x)$ as a product of terms of the form $(a + bx^c)^n$.

Use the binomial expansion for rational index, including only terms **up to x^2**.

Write out **the entire expression** before simplifying to reduce careless mistakes.

Choose 1 from the first bracket and multiply it with 1, $-nx$, and $\frac{n(n+1)}{2}x^2$ in the second bracket in turn. Do the same with the other 2 terms in the first bracket.

While doing so, discard any terms which have a power more than 2, since the expansion should be up to x^2.

Immediately before the binomial expansion was used, the expressions were $(1-x)^n$ and $(1+x)^{-n}$.

For the entire expansion to be valid, both binomial expansions involved must be valid, i.e. $|-x| < 1$ and $|x| < 1$, but both inequalities are equivalent to $|x| < 1$.

Let $x=\frac{1}{20}$ and $n=\frac{1}{4}$.

Since $|x|=\left|\frac{1}{20}\right|<1$, the expansion is valid.

$$\left(\frac{1-x}{1+x}\right)^n=\left(\frac{1-\frac{1}{20}}{1+\frac{1}{20}}\right)^{\frac{1}{4}}$$

$$=\left(\frac{20-1}{20+1}\right)^{\frac{1}{4}}$$

$$=\sqrt[4]{\frac{19}{21}}$$

$$\approx 1-2\left(\frac{1}{4}\right)\left(\frac{1}{20}\right)+2\left(\frac{1}{4}\right)^2\left(\frac{1}{20}\right)^2$$

$=\frac{3121}{3200}$, where $p=3121$ and $q=3200$. ■

$n=\frac{1}{4}$, since $\frac{19}{21}$ raised to the power $\frac{1}{4}$ is the fourth root of $\frac{19}{21}$. It remains to make

$$\frac{19}{21}=\frac{1-x}{1+x},$$

with $|x|<1$ for the expansion to be valid.

Solve for x directly to get $x=\frac{1}{20}$.

OR

Try to write $\frac{19}{21}$ in a similar form as $\frac{1-x}{1+x}$. You may come up with $\frac{20-1}{20+1}$, which suggests dividing both top and bottom by 20 to get

$$\frac{1-\frac{1}{20}}{1+\frac{1}{20}}$$

Hence, $x=\frac{1}{20}$.

Substitute $x=\frac{1}{20}$ back into the expansion. Since only the first few terms are taken, write " $\approx$ " instead of " $=$ " where the approximation occurs.

Vectors

When representing vectors as single letters, write a squiggle underneath: $\underset{\sim}{\mathbf{a}}$.

Basic Formulae

Distance between 2 points	If 2 points have position vectors $\mathbf{a}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}$ and $\mathbf{b}=\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$, then: $$\left|\overrightarrow{AB}\right|=\sqrt{(a_1-b_1)^2+(a_2-b_2)^2+(a_3-b_3)^2}$$	
Unit vector	A unit vector is a vector with modulus 1, usually used to specify direction. The unit vector $\hat{\mathbf{a}}$ of a vector $\mathbf{a}$ is given by: $$\hat{\mathbf{a}}=\frac{\mathbf{a}}{\lvert\mathbf{a}\rvert}$$	
Ratio theorem	If the point P divides AB such that $AP:PB=\lambda:\mu$, then: $$\overrightarrow{OP}=\frac{\mu\overrightarrow{OA}+\lambda\overrightarrow{OB}}{\lambda+\mu}$$ In particular, if P is the midpoint of AB, then $\overrightarrow{OP}=\frac{\overrightarrow{OA}+\overrightarrow{OB}}{2}$.	
Angle, θ, between:	2 vectors $\mathbf{a}$ and $\mathbf{b}$:	$\theta=\cos^{-1}\left(\frac{\mathbf{a}\cdot\mathbf{b}}{\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert}\right)$
	2 lines $l_1:\mathbf{r}=\mathbf{a}_1+\lambda\mathbf{b}_1$, $l_2:\mathbf{r}=\mathbf{a}_2+\mu\mathbf{b}_2$:	$\theta=\cos^{-1}\left(\frac{\lvert\mathbf{b}_1\cdot\mathbf{b}_2\rvert}{\lvert\mathbf{b}_1\rvert\lvert\mathbf{b}_2\rvert}\right)$
	2 planes $\pi_1:\mathbf{r}\cdot\mathbf{n}_1=p_1$, $\pi_2:\mathbf{r}\cdot\mathbf{n}_2=p_2$:	$\theta=\cos^{-1}\left(\frac{\lvert\mathbf{n}_1\cdot\mathbf{n}_2\rvert}{\lvert\mathbf{n}_1\rvert\lvert\mathbf{n}_2\rvert}\right)$
	a line $l:\mathbf{r}=\mathbf{a}+\lambda\mathbf{b}$ and a plane $\pi:\mathbf{r}\cdot\mathbf{n}=p$:	$\theta=\sin^{-1}\left(\frac{\lvert\mathbf{b}\cdot\mathbf{n}\rvert}{\lvert\mathbf{b}\rvert\lvert\mathbf{n}\rvert}\right)$
	Between 2 lines, 2 planes, or a line and a plane, there are generally **2 angles**, 1 acute and 1 obtuse. The modulus around the dot product in finds the **acute** angle. Between 2 vectors, there is only **1 unique angle**, thus there is no modulus around the dot product.	

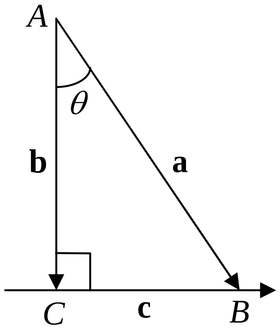
Equations of Lines & Planes

Line	**Vector**	**Cartesian**	
	$\mathbf{r}=\mathbf{a}+\lambda\mathbf{b},\ \lambda\in\mathbb{R}$ Make λ the subject for each of the 3 equations, then combine them to get the Cartesian form.	$\dfrac{x-a_1}{b_1}=\dfrac{y-a_2}{b_2}=\dfrac{z-a_3}{b_3}$ Use $\mathbf{a}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}$ and $\mathbf{b}=\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$ to get vector form.	
Plane	**Vector**	**Cartesian**	**Scalar Product**
	$\mathbf{r}=\mathbf{a}+\lambda\mathbf{b}+\mu\mathbf{c},\ \lambda,\mu\in\mathbb{R}$ Use $\mathbf{b}\times\mathbf{c}=\mathbf{n}$ to find a normal to the plane, then $\mathbf{a}\cdot\mathbf{n}=p$ to get the scalar product form.	$n_1x+n_2y+n_3z=p$ Set special conditions, e.g. $x=y=0$ to find 3 points A, P, Q on the plane. The vector form is then $\mathbf{r}=\overrightarrow{OA}+\lambda\overrightarrow{AP}+\mu\overrightarrow{AQ}$.	$\mathbf{r}\cdot\mathbf{n}=\mathbf{a}\cdot\mathbf{n}=p$ Expand the dot product to get the Cartesian equation.

Dot Product VS Cross Product

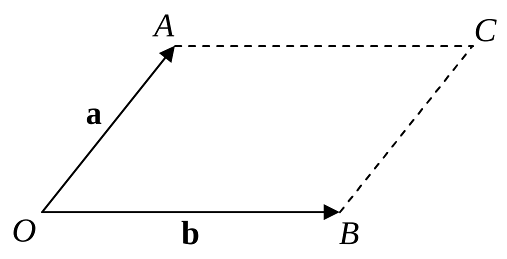
	Dot Product ($\mathbf{a}\cdot\mathbf{b}$)	**Cross Product ($\mathbf{a}\times\mathbf{b}$)**
Output	Scalar	Vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$
Formula	$\mathbf{a}\cdot\mathbf{b}=\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos\theta$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.	**Magnitude**: $\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\sin\theta$ **Direction**: Right-Hand Grip Rule
Calculation	$\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\cdot\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}=a_1b_1+a_2b_2+a_3b_3$ $\mathbf{a}\cdot\mathbf{a}=\lvert\mathbf{a}\rvert^2$	$\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\times\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}=\begin{pmatrix}a_2b_3-a_3b_2\\a_3b_1-a_1b_3\\a_1b_2-a_2b_1\end{pmatrix}$

Lengths & Areas



Say the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ form a right-angled triangle, as shown, and you know the vectors $\mathbf{a}$ and $\mathbf{b}$. You can then find the lengths AC and BC.
Let $\angle A=\theta$.

Length AC $=$ Length of projection of AB onto $\mathbf{b}$ $=AB\cos\theta$ $=\lvert\mathbf{a}\cdot\hat{\mathbf{b}}\rvert$	Length BC $=$ Perpendicular distance from B to $\mathbf{b}$ $=AB\sin\theta$ $=\lvert\mathbf{a}\times\hat{\mathbf{b}}\rvert$

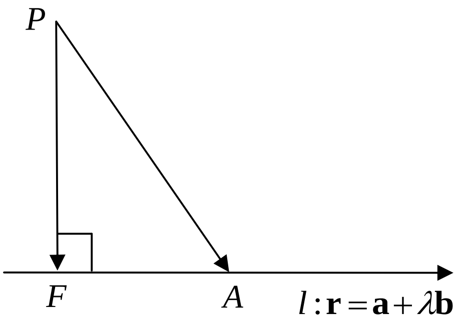


Let the points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively. Let C be the point such that $OACB$ is a parallelogram.

Area of triangle $OAB = \frac{1}{2}|\mathbf{a}\times\mathbf{b}|$

Area of parallelogram $OACB = |\mathbf{a}\times\mathbf{b}|$

Foot of Perpendicular & Perpendicular Distance



Say a point P has position vector $\mathbf{p}$, and a line l has equation $\mathbf{r}=\mathbf{a}+\lambda\mathbf{b}$, as shown. Say point A lies on l.

From P to l:

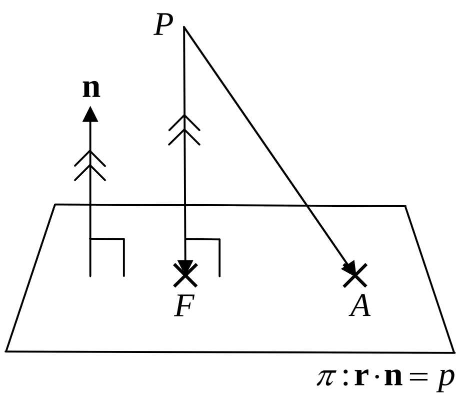
Foot of perpendicular, F, with position vector $\mathbf{f}$:

- Since F lies on l, $\mathbf{f}=\mathbf{a}+\lambda\mathbf{b}$ for some $\lambda\in\mathbb{R}$.
- Since $\overrightarrow{PF}$ and l are perpendicular, $(\mathbf{f}-\mathbf{p})\cdot\mathbf{b}=0$.
- $(\mathbf{a}+\lambda\mathbf{b}-\mathbf{p})\cdot\mathbf{b}=0$
- Solve for λ to find $\mathbf{f}$.

Perpendicular distance, $PF=\left|\overrightarrow{PF}\right|$:

Use any of these 3 methods.

1. Find $\mathbf{f}$ as above.
   $PF=|\mathbf{f}-\mathbf{p}|$
2. Find AP and AF.
   By Pythagoras' Theorem, $PF=\sqrt{AP^2-AF^2}$.
3. $PF=\left|\overrightarrow{AP}\times\hat{\mathbf{b}}\right|$



Say a point P has position vector $\mathbf{p}$, and a plane π has equation $\mathbf{r}\cdot\mathbf{n}=p$, as shown. Say point A lies on π.

From P to π:

Foot of perpendicular, F, with position vector $\mathbf{f}$:

- Since $\overrightarrow{PF}$ is perpendicular to π, $\overrightarrow{PF}$ is parallel to $\mathbf{n}$. Also, since P lies on $\overrightarrow{PF}$:
  $l_{PF}:\mathbf{r}=\mathbf{p}+\lambda\mathbf{n}$, $\lambda\in\mathbb{R}$.
- Since F lies on l_{PF}, $\mathbf{f}=\mathbf{p}+\lambda\mathbf{n}$ for some $\lambda\in\mathbb{R}$.
- Since F lies on π, $\mathbf{f}\cdot\mathbf{n}=p$
- $(\mathbf{p}+\lambda\mathbf{n})\cdot\mathbf{n}=p$
- Solve for λ to find $\mathbf{f}$.

Perpendicular distance, $PF=\left|\overrightarrow{PF}\right|$:

- $PF=\left|\overrightarrow{AP}\cdot\hat{\mathbf{n}}\right|$

SLE denotes system of linear equations. Use *PlySmlt2* on the GC to solve SLEs.

- **2 Vectors,** $\mathbf{a}$ and $\mathbf{b}$

Parallel:	The SLE given by $\mathbf{a} = \lambda\mathbf{b}$ has a non-zero real solution for λ.
Perpendicular:	$\mathbf{a} \cdot \mathbf{b} = 0$

- **2 Lines in 3 dimensions,** $l_1 : \mathbf{r} = \mathbf{a_1} + \lambda\mathbf{b_1}$ and $l_2 : \mathbf{r} = \mathbf{a_2} + \lambda\mathbf{b_2}$

Parallel:	$\mathbf{b_1}$ and $\mathbf{b_2}$ are parallel (see above).
Intersecting at a point:	• $\mathbf{b_1}$ and $\mathbf{b_2}$ are NOT parallel (see above), AND • The SLE given by $\mathbf{a_1} + \lambda\mathbf{b_1} = \mathbf{a_2} + \mu\mathbf{b_2}$ has a unique real solution for λ and μ. **Point of intersection,** P : If the solutions are λ_0 and μ_0, $\overrightarrow{OP} = \mathbf{a_1} + \lambda_0\mathbf{b_1}$ (or $\mathbf{a_2} + \mu_0\mathbf{b_2}$).
Skew:	• $\mathbf{b_1}$ and $\mathbf{b_2}$ are NOT parallel (see above), AND • The SLE given by $\mathbf{a_1} + \lambda\mathbf{b_1} = \mathbf{a_2} + \mu\mathbf{b_2}$ has no real solution for λ and μ.

- **3 Points,** A, B, C

Collinear (on the same line):	$\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{BC}$ are all parallel. You may pick **any 2** and show that they are parallel (see above).

- **Point,** Q, with position vector $\mathbf{q} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}$

Lies on a **line** $l : \mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$:	The SLE given by $\mathbf{q} = \mathbf{a} + \lambda\mathbf{b}$ has a real solution for λ.
Lies on a **plane** $\pi : \mathbf{r} \cdot \mathbf{n} = p$:	$\mathbf{q} \cdot \mathbf{n} = p$
Lies on a **plane** $\pi : \mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$:	The SLE given by $\mathbf{q} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$ has a real solution for λ and μ.
Lies on a **plane** $\pi : n_1x + n_2y + n_3z = p$:	$n_1q_1 + n_2q_2 + n_3q_3 = p$

- **Line & Plane,** $l:\mathbf{r}=\mathbf{a}+\lambda\mathbf{b}$ and $\pi:\mathbf{r}\cdot\mathbf{n}=p$

l lies on π:	$\mathbf{b}\cdot\mathbf{n}=0$ AND $\mathbf{a}\cdot\mathbf{n}=p$
l is parallel to π, but does not lie on π:	$\mathbf{b}\cdot\mathbf{n}=0$ AND $\mathbf{a}\cdot\mathbf{n}\neq p$
l intersects π at one point:	$\mathbf{b}\cdot\mathbf{n}\neq 0$ l is perpendicular to π if $\mathbf{b}$ and $\mathbf{n}$ are parallel (see above). **Point of intersection,** P: Solve $(\mathbf{a}+\lambda\mathbf{b})\cdot\mathbf{n}=p$ for λ. If the solution is λ_0, $\overrightarrow{OP}=\mathbf{a}+\lambda_0\mathbf{b}$

- **2 Planes,** $\pi_1:\mathbf{r}\cdot\mathbf{n}=p$ and $\pi_2:\mathbf{r}\cdot\mathbf{m}=q$, where $\mathbf{n}=\begin{pmatrix}n_1\\n_2\\n_3\end{pmatrix}$ and $\mathbf{m}=\begin{pmatrix}m_1\\m_2\\m_3\end{pmatrix}$

Parallel:	$\mathbf{n}$ and $\mathbf{m}$ are parallel (see above).
Intersecting along a line:	$\mathbf{n}$ and $\mathbf{m}$ are NOT parallel (see above) **Line of intersection,** l: Expand the dot products into Cartesian equations to obtain the SLE: $\begin{cases} n_1x+n_2y+n_3z=p \\ m_1x+m_2y+m_3z=q \end{cases}$ If *PlySmlt2* on the GC solves the SLE to give: $x_1=a_1+b_1x_3$ $x_2=a_2+b_2x_3$ $x_3=x_3$ where x_1, x_2, x_3 represent x, y, z respectively and a_1, a_2, b_1, b_2 are constants, $l=\begin{pmatrix}a_1\\a_2\\0\end{pmatrix}+\lambda\begin{pmatrix}b_1\\b_2\\1\end{pmatrix},\ \lambda\in\mathbb{R}$

- **<u>3 Planes</u>**

Use their Cartesian equations to obtain a SLE with 3 equations in 3 variables: x, y, z.

Do not intersect at any common point or line:	No real solution for x, y, z.
Intersect at one common point:	Unique solution for x, y, z. **Point of intersection,** P: If the solution to the SLE is x_0, y_0, z_0, $\overrightarrow{OP} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$
Intersect along one common line:	x and y can be solved in terms of z. **Line of intersection,** l: See above, where 2 planes intersect along a line.

<u>Example 1</u> [TJC06/Prelim/I/14]

The lines l_1 and l_2 have equations $\frac{x+8}{2} = \frac{5-y}{2} = z+1$ and $\mathbf{r} = \alpha \begin{pmatrix} 6 \\ -4 \\ 1 \end{pmatrix}$ respectively, where $\alpha \in \mathbb{R}$.

Find

(i) the acute angle between l_1 and the z-axis.
(ii) the position vector of the foot of the perpendicular, N, from the origin to l_1.
(iii) the point of intersection of l_1 and l_2.

Hence, find a vector equation of the reflection of line l_2 in the line l_1.

	Solution	**Comments**
(i)	Let $\lambda = \frac{x+8}{2} = \frac{5-y}{2} = z+1$.	Convert Cartesian equations of lines to vector equations, which are usually much more useful.

Then: $\begin{cases} x=-8+2\lambda \\ y=5-2\lambda \\ z=-1+\lambda \end{cases}$

Express x, y, z in terms of λ.

$l_1 : \mathbf{r} = \begin{pmatrix} -8 \\ 5 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix},\ \lambda \in \mathbb{R}$

z-axis: $\mathbf{r} = \beta \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},\ \beta \in \mathbb{R}$

z varies along the z-axis, while x and y do not, and the z-axis passes through the origin, explaining its vector equation.

Let θ be the acute angle between l_1 and the z-axis.

For the acute angle between 2 lines, use:

$$\theta = \cos^{-1}\left(\frac{|\mathbf{b_1}\cdot\mathbf{b_2}|}{|\mathbf{b_1}||\mathbf{b_2}|}\right)$$

$$\theta = \cos^{-1}\left[\frac{\left|\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}\cdot\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right|}{\left|\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}\right|\left|\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right|}\right] = 70.5° \text{ (1 d.p.)} \blacksquare$$

Since we are finding the **acute** angle, place a modulus around the dot product.

(ii) $\overrightarrow{ON} = \begin{pmatrix} -8+2\lambda \\ 5-2\lambda \\ -1+\lambda \end{pmatrix}$, for some $\lambda \in \mathbb{R}$.

(Since N lies on l_1)

$$\overrightarrow{ON}\cdot\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -8+2\lambda \\ 5-2\lambda \\ -1+\lambda \end{pmatrix}\cdot\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = 0$$

(Since $\overrightarrow{ON}$ is perpendicular to l_1)

$-16-10-1+4\lambda+4\lambda+\lambda=0$
$\lambda=3$

Expand the dot product, then solve for λ.

$$\overrightarrow{ON} = \begin{pmatrix} -8+2(3) \\ 5-2(3) \\ -1+3 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix} \blacksquare$$

Substitute λ into the original equation for $\overrightarrow{ON}$.

(iii) Let P be the point of intersection between l_1 and l_2.

Then $\overrightarrow{OP}=\begin{pmatrix}-8+2\lambda\\5-2\lambda\\-1+\lambda\end{pmatrix}=\begin{pmatrix}6\alpha\\-4\alpha\\\alpha\end{pmatrix}$,

for some $\lambda,\alpha\in\mathbb{R}$.

$$\begin{cases}2\lambda-6\alpha=8\\-2\lambda+4\alpha=-5\\\lambda-\alpha=1\end{cases}$$

Write the system of linear equations such that LHS has λ and α in that order, and RHS has a constant.

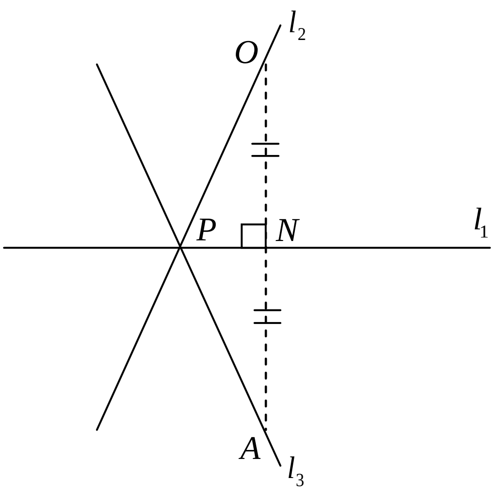
From the GC, $\lambda=-0.5$, $\alpha=-1.5$.

Then, solve using *PlySmlt2* on the GC.

$$\overrightarrow{OP}=\begin{pmatrix}6(-1.5)\\-4(-1.5)\\-1.5\end{pmatrix}=\begin{pmatrix}-9\\6\\-1.5\end{pmatrix}$$

Hence, P is $(-9,6,-1.5)$. ■

The question asks for "the point", so give the coordinates of the required point instead of its position vector.



Draw a diagram first to illustrate the question, and label the points clearly.

In a "Hence" question, use information from previous parts. With reference to the diagram, part (ii) gives $\overrightarrow{ON}$ and part (iii) gives $\overrightarrow{OP}$.

Let l_3 be the reflection of l_2 in l_1.

$$\overrightarrow{OA}=2\overrightarrow{ON}=\begin{pmatrix}-4\\-2\\4\end{pmatrix}$$

$$\overrightarrow{PA}=\begin{pmatrix}-4\\-2\\4\end{pmatrix}-\begin{pmatrix}-9\\6\\-1.5\end{pmatrix}=\begin{pmatrix}5\\-8\\5.5\end{pmatrix}$$

To obtain the vector equation of l_3, we can find 2 points which lie on l_3. 1 point is P, and part (ii) gives $\overrightarrow{OP}$.

From the diagram, we can use $\overrightarrow{ON}$ from part (ii) to find the position vector of A, another point on l_3, since $ON=AN$.

Now $\overrightarrow{PA}$ is a direction vector of l_3.

$$\overrightarrow{PA} = \frac{1}{2}\begin{pmatrix} 10 \\ -16 \\ 11 \end{pmatrix}$$

$$l_3 : \mathbf{r} = \begin{pmatrix} -4 \\ -2 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 10 \\ -16 \\ 11 \end{pmatrix},\ \mu \in \mathbb{R}$$ ■

When writing the equation of a line $l : \mathbf{r} = \mathbf{a} + \lambda \mathbf{b}$, choose the simplest vector for **a** (here, choose $\overrightarrow{OA}$ over $\overrightarrow{OP}$) and leave **b** in the simplest integer form. This makes your working easier in any later parts.

Example 2 [N08/I/11]

The equations of three planes p_1, p_2, p_3 are

$$2x - 5y + 3z = 3$$
$$3x + 2y - 5z = -5$$
$$5x + \lambda y + 17z = \mu$$

respectively, where λ and μ are constants. When $\lambda = -20.9$ and $\mu = 16.6$, find the coordinates of the point at which these planes meet.

The planes p_1 and p_2 intersect in a line l.

(i) Find a vector equation of l.
(ii) Given that all three planes meet in the line l, find λ and μ.
(iii) Given instead that the three planes have no point in common, what can be said about the values of λ and μ?
(iv) Find the Cartesian equation of the plane which contains l and the point $(1, -1, 3)$.

Solution	**Comments**
When $\lambda = -20.9$, $\mu = 16.6$,	Substitute the given values for λ and μ.
$\begin{cases} 2x - 5y + 3z = 3 \\ 3x + 2y - 5z = -5 \\ 5x - 20.9y + 17z = 16.6 \end{cases}$	Use the Cartesian equations of the 3 planes to obtain a SLE with 3 equations in 3 variables: x, y, z.
From the GC, $x = -\frac{4}{11}$, $y = -\frac{4}{11}$, $z = \frac{7}{11}$	Solve it using GC, then the unique solution for x, y, z gives the coordinates of the point of intersection.
Coordinates of the point at which p_1, p_2 and p_3 meet: $\left(-\frac{4}{11}, -\frac{4}{11}, \frac{7}{11}\right)$ ■	Give the coordinates of the point, not its position vector.

(i) $\begin{cases} 2x-5y+3z=3 \\ 3x+2y-5z=-5 \end{cases}$

From the GC,

$\begin{cases} x=-1+\alpha \\ y=-1+\alpha \\ z=\alpha \end{cases}$, where $\alpha \in \mathbb{R}$.

$$l:\mathbf{r}=\begin{pmatrix}-1\\-1\\0\end{pmatrix}+\alpha\begin{pmatrix}1\\1\\1\end{pmatrix},\ \alpha \in \mathbb{R}\ \blacksquare$$

Use the Cartesian equations of the 2 planes to obtain a SLE with 2 equations in 3 variables: x, y, z.

Solve it using GC. If the GC gives the solutions for x and y in terms of z, the 2 planes intersect along a line.

Write z as a parameter, say α, to get the vector equation of the line.

(ii) $$p_3:\mathbf{r}\cdot\begin{pmatrix}5\\\lambda\\17\end{pmatrix}=\mu$$

Since l lies on p_3, every point which lies on l also lies on p_3.

$$\left[\begin{pmatrix}-1\\-1\\0\end{pmatrix}+\alpha\begin{pmatrix}1\\1\\1\end{pmatrix}\right]\cdot\begin{pmatrix}5\\\lambda\\17\end{pmatrix}=\mu,\text{ for all } \alpha \in \mathbb{R}.$$

$-5+5\alpha-\lambda+\lambda\alpha+0+17\alpha=\mu$

$(\lambda+22)\alpha-5-\lambda=\mu$

p_1 and p_2 already intersect in the line l, with equation found in part (i). If all 3 planes intersect in l, l must lie on p_3.

The method shown here uses the fact that for each value of $\alpha \in \mathbb{R}$,

$$\mathbf{s}=\begin{pmatrix}-1\\-1\\0\end{pmatrix}+\alpha\begin{pmatrix}1\\1\\1\end{pmatrix}$$

gives the position vector of a point on l.

Since the above equation is true for all $\alpha \in \mathbb{R}$, comparing coefficients,

$\begin{cases} \lambda+22=0 \\ -5-\lambda=\mu \end{cases}$

$\lambda=-22$

$\mu=\lambda+5=-17$

$\lambda=-22$, $\mu=-17$ $\blacksquare$

Consider $(\lambda+22)\alpha-5-\lambda=\mu$ to have α as variable, with λ and μ constant. Since every point on l lies on p_3, every value of $\alpha \in \mathbb{R}$ fulfils the equation.

(iii) p_1, p_2 and p_3 have no point in common if and only if l is parallel to p_3 and l does not intersect p_3.

p_1 and p_2 already intersect in the line l.

Case 1: l is not parallel to p_3, then l and p_3 will intersect. This intersection point will be a point in common to p_1, p_2 and p_3.

Case 2: l lies on p_3, then any point on l will be a point in common to p_1, p_2 and p_3.

Case 3: l is parallel to p_3 and l does not intersect p_3, then since p_1 and p_2 only intersect in l, p_1, p_2 and p_3 have no point in common.

l is parallel to p_3 if and only if:

$$\begin{pmatrix}1\\1\\1\end{pmatrix}\cdot\begin{pmatrix}5\\\lambda\\17\end{pmatrix}=0$$

$$5+\lambda+17=0$$
$$\lambda=-22$$

l is parallel to p_3 if and only if the direction vector of l is perpendicular to the normal to p_3, i.e. their dot product is 0.

When l is parallel to p_3,
l intersects p_3 if and only if:

$$\begin{pmatrix}-1\\-1\\0\end{pmatrix}\cdot\begin{pmatrix}5\\\lambda\\17\end{pmatrix}=\mu$$

$$\mu=-\lambda-5$$

When l is parallel to p_3, either **every point** on l or **no point** on l lies on p_3. Hence, it is enough to check if **a point** on l lies on p_3.

Hence, both $\lambda=-22$ and $\mu\neq-\lambda-5$ must be true.
$\lambda=-22$ and $\mu\neq-17$ ■

This corresponds to the situation when l is parallel to p_3 and l does not intersect p_3.

(iv) Let π be the required plane.

$$\begin{pmatrix}-1\\-1\\0\end{pmatrix}-\begin{pmatrix}1\\-1\\3\end{pmatrix}=\begin{pmatrix}-2\\0\\-3\end{pmatrix}$$

$$\begin{pmatrix}1\\1\\1\end{pmatrix}\times\begin{pmatrix}-2\\0\\-3\end{pmatrix}=\begin{pmatrix}-3\\1\\2\end{pmatrix}$$

$$p_4:\mathbf{r}\cdot\begin{pmatrix}-3\\1\\2\end{pmatrix}=\begin{pmatrix}1\\-1\\3\end{pmatrix}\cdot\begin{pmatrix}-3\\1\\2\end{pmatrix}=2$$

$$\begin{pmatrix}x\\y\\z\end{pmatrix}\cdot\begin{pmatrix}-3\\1\\2\end{pmatrix}=2$$

Cartesian equation of π:
$-3x+y+2z=2$ ■

To get the scalar product equation of the plane π in the form $\mathbf{r}\cdot\mathbf{n}=p$,

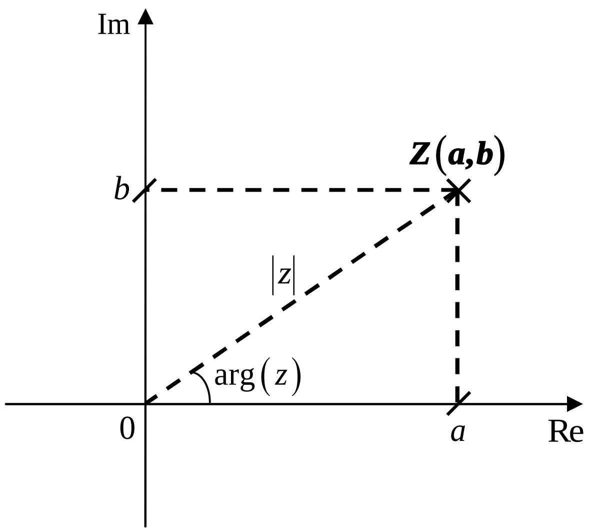
1. Find the normal **n** by taking the cross product of 2 vectors which lie on the plane.

   Since l lies on π, the direction vector of l also lies on π. Use a point which lies on l and the given point $(1,-1,3)$ to find another vector which lies on π.

2. Once **n** is known, find the constant p using
   $$\mathbf{r}\cdot\mathbf{n}=\mathbf{a}\cdot\mathbf{n}=p$$
   where **a** is the position vector of any point on π.

Expand the dot product to get the Cartesian equation of π, as required.

Complex Numbers



A **complex number** in Cartesian form:

$$z = a + bi$$

consists of a **real** part a and an **imaginary** part bi, where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$.

Any complex number $z = a + bi$ can be represented geometrically as **a point** Z on the complex plane.

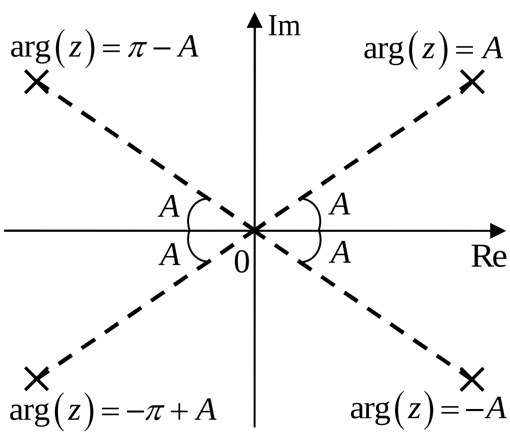
On an Argand diagram, the horizontal axis is the real axis, and the vertical axis is the imaginary axis.

Let $z = a + bi$, $a, b \in \mathbb{R}$ be a complex number in Cartesian form.
On an Argand diagram, let Z and Z^* be the points representing z and z^* respectively.

- **Real part, $\mathrm{Re}(z)$ and imaginary part, $\mathrm{Im}(z)$**
  - $\mathrm{Re}(z) = a$
  - $\mathrm{Im}(z) = b$ (not bi)
  - $\mathrm{Re}(z)$ is the x-coordinate of Z
  - $\mathrm{Im}(z)$ is the y-coordinate of Z

- **Modulus, $|z|$**
  - $|z| = \sqrt{a^2 + b^2}$
  - Distance of Z from the origin (length of OZ)

- **Argument, $\arg(z)$**
  - Angle that $\overrightarrow{OZ}$ makes with the positive real axis, in an anticlockwise sense
  - $-\pi < \arg(z) \le \pi$
  - $\arg(0)$ is undefined
  - Calculate the basic angle, $A = \tan^{-1}\left|\frac{b}{a}\right|$
  - Draw the Argand diagram: $\arg(z)$ depends on both A and the quadrant which the point Z is in.



- **Conjugate, $z^* = a - bi$**
  - $\operatorname{Re}(z^*) = \operatorname{Re}(z)$
  - $\operatorname{Im}(z^*) = \operatorname{Im}(z)$
  - $|z^*| = |z|$
  - $\arg(z^*) = -\arg(z)$
  - $zz^* = |z|^2$
  - $z + z^* = 2\operatorname{Re}(z)$
  - $z - z^* = 2i\operatorname{Im}(z)$
  - The conjugate distributes over addition, subtraction, multiplication, and division:
    - $(zw)^* = z^* w^*$, $\left(\frac{z}{w}\right)^* = \frac{z^*}{w^*}$, $(z+w)^* = z^* + w^*$, $(z-w)^* = z^* - w^*$
  - To simplify a fraction with a complex number in its denominator, multiply both numerator and denominator by the **complex conjugate** of the denominator.
  - Z is Z^* reflected about the x-axis.

Forms of Complex Numbers

- **Cartesian form**
  - $\boldsymbol{z = a + bi}$
  - Useful for addition and subtraction of complex numbers

- **Polar form** (also Trigonometric form or Modulus-Argument form)
  - $\boldsymbol{z = r(\cos\theta + i\sin\theta)}$
  - Convert from polar to Cartesian or exponential forms, which are more useful.

- **Exponential form**
  - $\boldsymbol{z = re^{i\theta}}$
  - Useful for multiplication, division and exponentiation of complex numbers

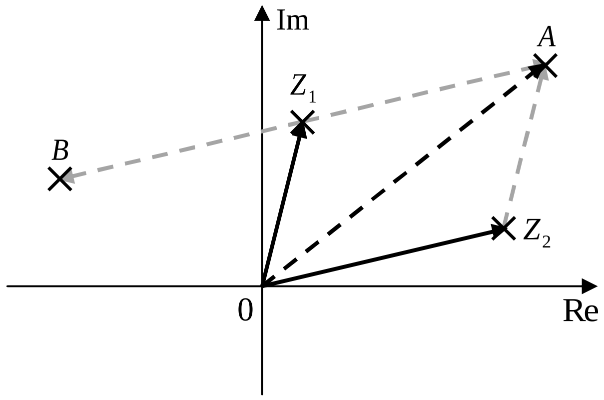
- If $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$, i.e. z is in Polar or Exponential form,
  - $|z| = r$
  - $\arg(z) = \theta$
  - $z^* = re^{-i\theta} = r(\cos\theta - i\sin\theta)$

- Conversion between forms
  - Cartesian to Polar or Exponential: $r = |z|$, $\theta = \arg(z)$
  - Polar or Exponential to Cartesian: $a = r\cos\theta$, $b = r\sin\theta$
  - Between Polar and Exponential: Use the same r and θ

Operations on Complex Numbers

Let $z_1 = a_1 + b_1 i = r_1 e^{i\theta_1}$ and $z_2 = a_2 + b_2 i = r_2 e^{i\theta_2}$.
On an Argand diagram, let Z_1 and Z_2 be the points representing z_1 and z_2 respectively.

- **Addition & Subtraction**



  - $z_1 + z_2 = (a_1 + a_2) + (b_1 + b_2)i$
  - $z_1 - z_2 = (a_1 - a_2) + (b_1 - b_2)i$
  - Complex numbers can be represented as vectors on an Argand diagram. Hence, addition and subtraction of complex numbers can be represented as addition and subtraction of vectors.
  - $\overrightarrow{OZ_1} + \overrightarrow{OZ_2} = \overrightarrow{OA}$, the position vector of the point representing $z_1 + z_2$.
  - $\overrightarrow{OZ_1} - \overrightarrow{OZ_2} = \overrightarrow{OB}$, the position vector of the point representing $z_1 - z_2$.

- **Multiplication & Division**
  - $|z_1 z_2| = |z_1||z_2| = r_1 r_2$
  - $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|} = \dfrac{r_1}{r_2}$
  - $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) = \theta_1 + \theta_2$
  - $\arg\left(\dfrac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) = \theta_1 - \theta_2$
  - Starting from $\overrightarrow{OZ_1}$,
    - An **anticlockwise** pivot by θ_2 radians about the origin and scaling by a factor of r_2 gives the position vector of the point representing $z_1 z_2$.
    - A **clockwise** pivot by θ_2 radians about the origin and scaling by a factor of r_2 gives the position vector of the point representing $\dfrac{z_1}{z_2}$.
    - An **anticlockwise** pivot by θ radians about the origin gives the position vector of the point representing $z_1 e^{i\theta}$. In particular, when $\theta = \dfrac{\pi}{2}$, the position vector of the point representing iz_1 is given.

Complex Roots of Polynomial Equations

Non-real complex roots of a polynomial equation with **real coefficients** occur in conjugate pairs. If α is a root of a polynomial equation with real coefficients, then α^* is also a root of the equation.

> **Example**: Since $z^3+5z^2+17z+13=0$ is a polynomial equation with real coefficients and $z=\alpha=-2+3i$ is a root, $z=\alpha^*=-2-3i$ is also a root. Equivalently, since $z-2-3i$ is a factor of the polynomial, $z-2+3i$ is also a factor.

De Moivre's Theorem

If $z=re^{i\theta}=r(\cos\theta+i\sin\theta)$,

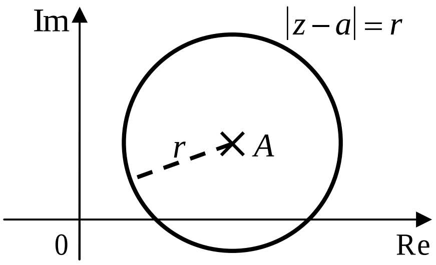
$z^n=r^ne^{i(n\theta)}=r^n\left[\cos(n\theta)+i\sin(n\theta)\right]$

Loci of Complex Numbers

A **locus** (plural **loci**) is a set of points which fulfill a given condition.

On an Argand diagram, let Z, A, B be the points representing z, a, b respectively.
Z is a **variable point** which can move, while A and B are **fixed points**.

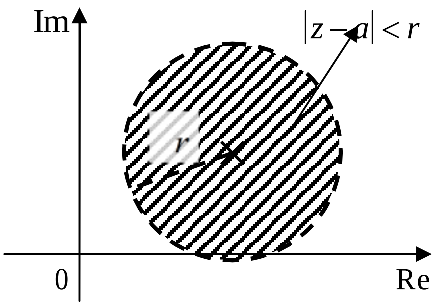
- $|z-a|$ is the **distance** between Z and A.
- $\arg(z-a)$ is the **angle** made by the vector $\overrightarrow{AZ}$ with the positive real axis.



$\boldsymbol{|z-a|=r}$

What it means:
The distance between Z and A is a **constant**, r

Locus of Z :
A **circle** with centre A and radius r

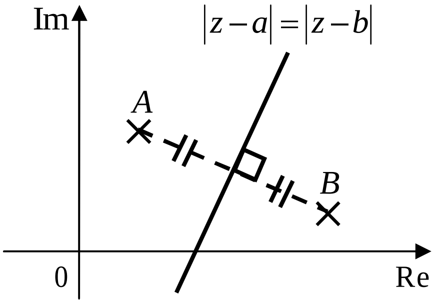


$\boldsymbol{|z-a|<r}$
The distance between Z and A is less than r.

Locus of Z :
The area inside the circle with centre A and radius r.

$\boldsymbol{|z-a|>r}$
The distance between Z and A is more than r.

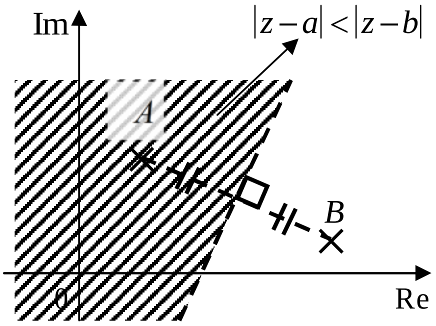
Locus of Z :
The area outside the circle with centre A and radius r.



$|z-a|=|z-b|$

What it means:
Z is an **equal distance** from each of A and B

Locus of Z :
The **perpendicular bisector** of the line joining A and B

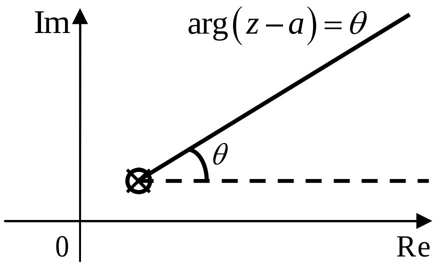


$|z-a|<|z-b|$
Z is closer to A than it is to B.

$|z-a|>|z-b|$
Z is closer to B than it is to A.

Locus of Z in each case:
$|z-a|=|z-b|$ separates the plane into 2 areas:

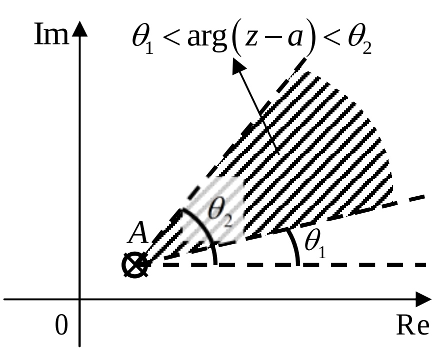
- $|z-a|<|z-b|$ is the area containing A.
- $|z-a|>|z-b|$ is the area containing B.



$\arg(z-a)=\theta$

What it means:
The angle made by $\overrightarrow{AZ}$ with the positive real axis is a **constant**, θ

Locus of Z :
The **half-line** starting at (but not including) A and making an angle θ with the positive real axis



$\theta_1<\arg(z-a)<\theta_2$
The angle made by $\overrightarrow{AZ}$ with the positive real axis is between θ_1 and θ_2.

Locus of Z :
The area bounded from the locus $\arg(z-a)=\theta_1$, anticlockwise to the locus $\arg(z-a)=\theta_2$.

FAQ

Q: Find the roots of the equation $(z-\alpha)^n = w$, where z and w are complex numbers, α is a constant, and n is a positive integer.

A:
1. Convert w to exponential form. Say $w = re^{i\theta}$.
$$(z-\alpha)^n = re^{i\theta}$$
2. Introduce $2k\pi$ to θ.
$$(z-\alpha)^n = re^{(\theta+2k\pi)i}\text{, where } k \in \mathbb{Z}.$$
3. Raise both sides to the power of $\frac{1}{n}$.
$$z-\alpha = r^{\frac{1}{n}}\left[e^{(\theta+2k\pi)i}\right]^{\frac{1}{n}}$$
4. By De Moivre's Theorem,
$$z-\alpha = r^{\frac{1}{n}}e^{\left(\frac{\theta+2k\pi}{n}\right)i}$$
5. Solve for z.
$$z = \alpha + r^{\frac{1}{n}}e^{\left(\frac{\theta+2k\pi}{n}\right)i}$$
6. Write the possible values of k. In general, $k = 0, 1, 2, \ldots, n-1$.

   However, if the roots must be given in the form $re^{i\theta}$ with $-\pi < \theta \le \pi$, substitute values of k to **check if θ is in the required range**. There will be n consecutive integer values of k.

   Reject those values of k which make any expression **undefined** (denominator 0).

Q: Solve for z and w in 2 simultaneous linear equations.

Example [PJC07/Prelim/I/1]

Solve the simultaneous equations
$$iw + z = -1 - i \text{ ----- (A)}$$
$$2z - (1+i)w = \frac{40}{6-2i} \text{ ----- (B)}$$
giving each answer in the form $a+bi$, where a and b are real.

A: Method 1
1. Multiply (A) by 2 such that the coefficient of z in (A) and (B) are both 2.
2. Use (A) – (B) to eliminate the $2z$ term from both equations.
3. Solve for w, then solve for z.

Method 2
1. Using (A), express z in terms of w: $z = -1 - i - iw$.
2. Substitute z in (B) with the expression in terms of w.
3. Solve for w, then solve for z.

Q: Find the complex (real and non-real) roots of a polynomial equation.

A:
- If the polynomial equation in z has degree n (the highest power of z is n), and the coefficient of z^n is k, the equation can be expressed as a product of k and n linear factors of the form $z-\alpha$.

- If α **is a root** of the equation, $z-\alpha$ **is a factor** of the polynomial.

1. If the polynomial equation has **real coefficients**,
   Given that α is a non-real complex root of the equation, α^* is also a root.

2. Using any given roots or factors, **long division** or **comparing coefficients**, express the polynomial as a product of its factors.

3. Use the **general quadratic formula** to factorize any quadratic factors.

4. When the equation is in this form with n linear factors of the form $z-\alpha$:
   $$k(z-\alpha_1)(z-\alpha_2)\ldots(z-\alpha_n)=0$$
   then $\alpha_1,\alpha_2,\ldots,\alpha_n$ are the n (possibly repeated) roots of the equation.

Q: Sketch loci on an Argand diagram.

A:
- Draw all circles, lengths and angles **accurately** and **to scale** using a compass, ruler and protractor. Ensure the real and imaginary axes are to the same scale.

- For loci of the form $\arg(z-a)=\theta$, draw a **small circle** at the point representing a to show that it is excluded, since $\arg(a-a)=\arg(0)$ is undefined.

- For **strict** inequalities, draw the loci of the equality cases as **dashed** to show that they are excluded, e.g. for $|z-a|<r$, draw the locus $|z-a|=r$ as a dashed circle.

- Label all loci with their equations, and label required regions/sets of points.

Q: Find the minimum/maximum value of $|z-a|$, $\arg(z-a)$ or $|z-w|$,
OR Find the points of intersection of the loci of points representing z and w, where z and w are variable while a is fixed.

A:
- Interpret what expressions mean **geometrically** on the Argand diagram, rather than interpret them algebraically. e.g. minimum value of $|z-w|$ is the minimum distance between any point on the locus of points representing z and any point on the locus of points representing w.

- As a last resort for finding points of intersection, find the Cartesian equations of the lines or circles and solve the equations simultaneously.

- Find lengths using the formula for the **distance between 2 points**.

- Find **right-angled triangles** to use **trigonometric ratios**.
- If the locus of z is a circle, $|z-b|=r$ and the fixed complex number a is represented by the point A outside the circle,
  - Use **tangents** to the circle to find the min./max. values of $\arg(z-a)$.
  - Draw a line from point A **through the centre** of the circle. The line and circle will have 2 points of intersection. Use the distance from A to these 2 points to find the min./max. values of $|z-a|$.

Example 1 [SRJC08/Prelim/II/3]

(i) Show that $\dfrac{1+e^{i\theta}}{1-e^{i\theta}}$ can be reduced to $i\cot\left(\dfrac{\theta}{2}\right)$.

(ii) Find all the roots of the equation $z^8=\dfrac{1}{\sqrt{2}}(1-i)$, giving each root in the form $re^{i\theta}$ where $r>0$ and $-\pi<\theta\le\pi$.

(iii) Hence deduce all the roots of $\left(\dfrac{w-1}{w+1}\right)^8=\dfrac{1}{\sqrt{2}}(1-i)$.

Solution	**Comments**
(i) $\dfrac{1+e^{i\theta}}{1-e^{i\theta}}=\dfrac{e^{\left(-\frac{\theta}{2}\right)i}+e^{\left(\frac{\theta}{2}\right)i}}{e^{\left(-\frac{\theta}{2}\right)i}-e^{\left(\frac{\theta}{2}\right)i}}$	"Factoring out $\dfrac{\theta}{2}$" is a common theme in Complex Numbers proving questions.
$=\dfrac{2\,\mathrm{Re}\left[e^{\left(-\frac{\theta}{2}\right)i}\right]}{2i\ \mathrm{Im}\left[e^{\left(-\frac{\theta}{2}\right)i}\right]}$	Here, it takes the form of factoring out $e^{\left(\frac{\theta}{2}\right)i}$ from both the numerator and denominator. The aim is to create expressions of the form $z+z^*$ or $z-z^*$, which can be written as $\mathrm{Re}(z)$ or $2i\,\mathrm{Im}(z)$ respectively.
$=\dfrac{2\cos\left(-\frac{\theta}{2}\right)}{2i\sin\left(-\frac{\theta}{2}\right)}$	
$=\dfrac{\cos\left(\frac{\theta}{2}\right)}{-i\sin\left(\frac{\theta}{2}\right)}$	$\cos(-\theta)=\cos\theta$ $\sin(-\theta)=-\sin\theta$

$$=\left(-\frac{1}{i}\right)\cot\left(\frac{\theta}{2}\right)$$

$$=i\cot\left(\frac{\theta}{2}\right)$$

$$-\frac{1}{i}=-\frac{i}{i^2}=-\frac{i}{-1}=i$$

(ii)

$$z^8=\frac{1}{\sqrt{2}}(1-i)=e^{\left(-\frac{\pi}{4}\right)i}$$

$$=e^{\left(\frac{8k-1}{4}\right)\pi i}$$

1. Convert $\frac{1}{\sqrt{2}}(1-i)$ to exponential form.

$$=e^{\left(2k\pi-\frac{\pi}{4}\right)i}\text{, where } k\in\mathbb{Z}.$$

2. Introduce $2k\pi$ to $-\frac{\pi}{4}$.

By De Moivre's Theorem,

$$z=e^{\left(\frac{8k-1}{32}\right)\pi i},$$

3. Raise both sides to the power of $\frac{1}{8}$, and
4. Use De Moivre's Theorem.

where $k=-3,-2,-1,0,1,2,4$ [redacted]

5. Write out the possible values of k. Since the question requires $-\pi<\theta\le\pi$, there will be exactly 8 consecutive integer values of k which will give θ in the required range.

   Substitute values of k to check which values are valid.

(iii)

$$\left(\frac{w-1}{w+1}\right)^8=\frac{1}{\sqrt{2}}(1-i)$$

Let $\theta=\left(\frac{8k-1}{32}\right)\pi$.

$$z=e^{i\theta}=\frac{w-1}{w+1}$$

$$zw+z=w-1$$

$$1+z=w(1-z)$$

$$w=\frac{1+z}{1-z}=\frac{1+e^{i\theta}}{1-e^{i\theta}}$$

Since $\frac{w-1}{w+1}$ corresponds with z in part (ii), equate the two. To deduce the roots of the equation in w, express w in terms of z.

$$w=i\cot\left(\frac{\theta}{2}\right)=i\cot\left[\left(\frac{8k-1}{64}\right)\pi\right],$$

where $k=-3,-2,-1,0,1,2,4$ [redacted]

k takes the same values as in part (ii), since the values of k are for the roots of the equation in z from part (ii).

Example 2 [AJC08/Prelim/I/6]

If $z=i$ is a root of the equation $z^3+(1-3i)z^2-(2+3i)z-2=0$, determine the other roots. Hence find the roots of the equation $w^3+(1+3i)w^2+(3i-2)w-2=0$.

Solution

Let $z^3+(1-3i)z^2-(2+3i)z-2$
$=(z-i)(z^2+az+b)$,
for some complex numbers a and b.

$z^3+(1-3i)z^2-(2+3i)z-2$
$=z^3+az^2+bz-iz^2-aiz-bi$
$=z^3+(a-i)z^2+(b-ai)z-bi$

Comparing coefficients,
$1-3i=a-i \Rightarrow a=1-2i$
$-2=-bi \Rightarrow b=\dfrac{2}{i}=-2i$
$z^2+az+b=z^2+(1-2i)z-2i$

When $z^2+(1-2i)z-2i=0$,

$$z=\frac{-(1-2i)\pm\sqrt{(1-2i)^2-4(1)(-2i)}}{2(1)}$$
$$=\frac{2i-1\pm\sqrt{-3+4i}}{2}$$
$$=2i \text{ or } -1$$

Hence, the other roots are $2i$ and -1. ■

Taking conjugates on both sides,
$\left[z^3+(1-3i)z^2-(2+3i)z-2\right]^*=0^*$

Comments

Let $P(z)=z^3+(1-3i)z^2-(2+3i)z-2$.

Not all of the coefficients of $P(z)$ are real. Hence, though i is a root, $(i)^*=-i$ may not be a root.

After $P(z)$ (degree 3) is divided by $z-i$ (degree 2), it should leave a quadratic factor (degree 2). Expand $(z-i)(z^2+az+b)$ and compare the coefficients.

Do NOT write $P(z)$ in terms of linear factors first: $(z-i)(z-a)(z-b)$, since it will be difficult to compare coefficients.

Alternative method

Use long division to find the remainder when $P(z)$ is divided by $z-i$.

Use the general quadratic formula to factorize the quadratic factor.

Use the GC to perform calculations with complex numbers, including square roots, to avoid careless mistakes.

Let $Q(w)=w^3+(1+3i)w^2+(3i-2)w-2$.

Observe that the coefficients of w in $Q(w)$ are the conjugates of the coefficients of z in $P(z)$. Hence, convert the coefficients of z in $P(z)$ to their conjugates.

$$\left(z^3\right)^* + (1-3i)^*\left(z^2\right)^* - (2+3i)^* z^* - 2 = 0$$
$$\left(z^*\right)^3 + (1+3i)\left(z^*\right)^2 + (3i-2) z^* - 2 = 0$$

Hence, the roots of
$w^3 + (1+3i)w^2 + (3i-2)w - 2 = 0$
are the conjugates of the roots of
$z^3 + (1-3i)z^2 - (2+3i)z - 2 = 0$,
i.e. i, $-2i$, and -1. ■

Simplify the LHS using $(z \pm w)^* = z^* \pm w^*$, and $(zw)^* = z^* w^*$, which extends to:

$$\left(z^3\right)^* = (z \cdot z \cdot z)^* = z^* \cdot z^* \cdot z^* = \left(z^*\right)^3$$

Since the equation in z^* is now similar to the equation in w, each root of $Q(w)$ corresponds to the conjugate of a root of $P(z)$.

Example 3 [AJC07/Prelim/II/3]

In an Argand diagram, the point A represents the fixed complex number a, where $0 < \arg(a) < \frac{\pi}{2}$. The complex numbers z and w are such that

$$|z - 2ia| = |a| \text{ and } |w| = |w + ia|$$

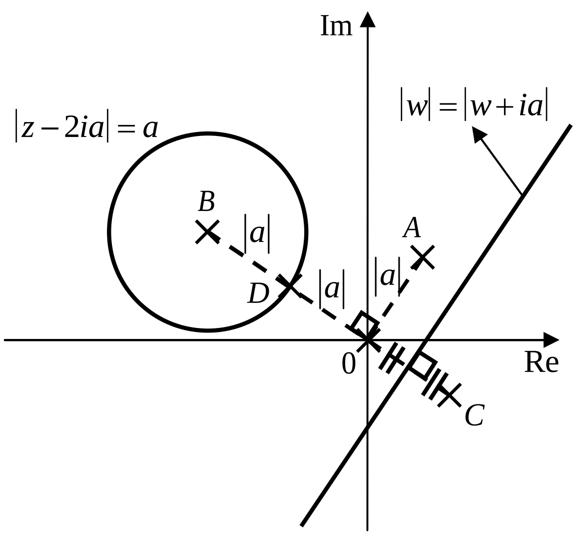
Sketch, in a single diagram, the loci of the points representing z and w.

Find:

(i) the minimum value of $|z - w|$ in terms of $|a|$,

(ii) the range of values of $\arg\left(\frac{1}{z}\right)$ in terms of $\arg(a)$.

Solution



Comments

Let B be the point representing $2ia$.
Let C be the point representing $-ia$.

If $0 < \arg(a) < \frac{\pi}{2}$, A is in the first quadrant of the complex plane. Plot any such point A.

The locus of points representing z is a circle centred at B with radius $|a|$.

The locus of points representing w is the perpendicular bisector of the line segment joining the origin and C.

From $\overrightarrow{OA}$, an anticlockwise pivot about the origin by $\frac{\pi}{2}$ radians and a stretching by factor 2 gives $\overrightarrow{OB}$.

From $\overrightarrow{OA}$, a clockwise pivot about the origin by $\frac{\pi}{2}$ radians gives $\overrightarrow{OC}$.

(i) Minimum value of $|z-w|$

$= OD + \frac{1}{2}OC$

$= |a| + \frac{1}{2}|a|$

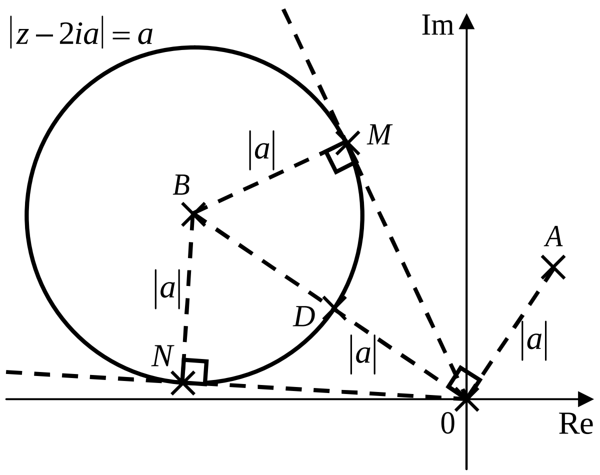
$= \frac{3}{2}|a|$ ■

Geometrically, the minimum value of $|z-w|$ is the minimum distance between a point on the locus of points representing Z and a point on the locus of points representing W.

Since the locus of points representing w is a bisector of OC, it divides OC into 2 segments of equal length, $\frac{1}{2}|a|$.

(ii) $\arg\left(\frac{1}{z}\right) = \arg(1) - \arg(z) = -\arg(z)$

Express the condition in terms of a known variable, $\arg(z)$.



Do not redraw the diagram, a relevant part of it is copied here for easier reference.

Let M and N be the points at which the tangent to the locus of z passes through the origin, as shown.

Let M and N represent the complex numbers m and n respectively.

Then $\arg(m) \le \arg(z) \le \arg(n)$.

Using **tangents to the circle** is a common technique to find the min./max. value of $\arg(z-a)$ when the locus of points representing z is a circle.

If Z is the point representing z, $\arg(z)$ is the **angle** made by the vector $\overrightarrow{OZ}$ with the positive real axis.

$\angle BOM = \angle BON = \sin^{-1}\left(\frac{|a|}{2|a|}\right) = \frac{\pi}{6}$

Use $\arg(a)$ since the question requires the final answer to be in terms of $\arg(a)$.

$$\arg(m) = \arg(a) + \frac{\pi}{2} - \angle BOM = \arg(a) + \frac{\pi}{2} - \frac{\pi}{6} = \arg(a) + \frac{\pi}{3}$$

The line OB passes through the centre of the circle. Hence, use the angle OB makes with the positive real axis as a reference, and add/subtract from it.

$$\arg(n) = \arg(a) + \frac{\pi}{2} + \angle BON = \arg(a) + \frac{\pi}{2} + \frac{\pi}{6} = \arg(a) + \frac{2\pi}{3}$$

$\overrightarrow{OA}$ makes an angle of $\arg(a)$ with the positive real axis, and $\angle BOA = \frac{\pi}{2}$. Hence, $\overrightarrow{OB}$ makes an angle of $\arg(a) + \frac{\pi}{2}$ with the positive real axis.

$$\arg(a) + \frac{\pi}{3} \le \arg(z) \le \arg(a) + \frac{2\pi}{3}$$

Convert the range of values of $\arg(z)$ to the range of values of $\arg\left(\frac{1}{z}\right)$.

$$-\arg(a) - \frac{\pi}{3} \ge -\arg(z) \ge -\arg(a) - \frac{2\pi}{3}$$

$$-\arg(a) - \frac{2\pi}{3} \le \arg\left(\frac{1}{z}\right) \le -\arg(a) - \frac{\pi}{3} \quad \blacksquare$$

Multiply each term in the inequality by -1, and flip the inequality signs.

Example 4 [NJC08/Prelim/I/6]

(i) The set of points P in an Argand diagram represents the complex number z that satisfies $|z-2-i| = 2$. Sketch the locus of P.

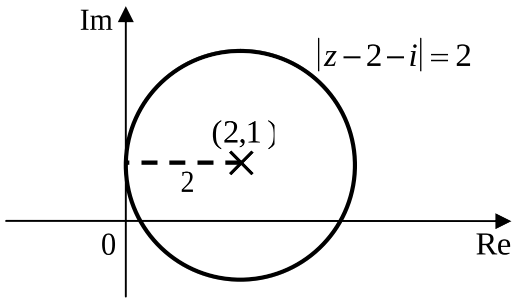
(ii) The set of points Q represents another complex number w given by $\arg(w-2+3i) = \theta$ where $-\pi < \theta \le \pi$. Give a geometrical description of the locus of Q.

(a) Find the range of values of θ such that the locus of Q meets the locus of P more than once.

(b) In the case where $\theta = \tan^{-1}\left(\frac{3}{4}\right)$, find the least value of $|z-w|$ in exact form.

Solution | **Comments**

(i)



$$\left|z-(2+i)\right|=2$$

The locus of P is a circle centred at $(2,1)$ with radius 2.

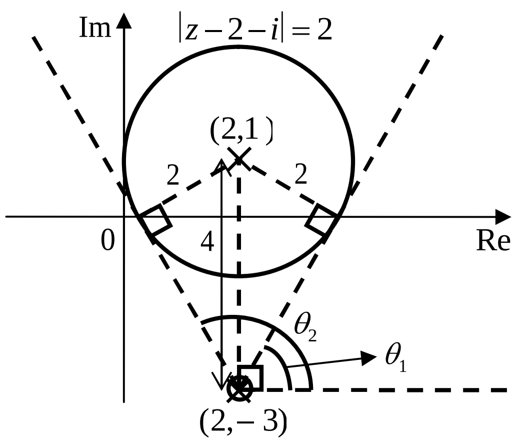
Show on the diagram that the circle is tangent to the imaginary axis. Examiners will look out for the relative position of lines and circles to the axes.

(ii) The locus of Q is a half-line starting at (but not including) $(2,-3)$ and making an angle θ with the positive real axis.

$$\arg\left[w-(2-3i)\right]=\theta$$

Remember to write "(but not including)".

(ii)(a)



Using tangents to the circle is the same technique as in Example 3, part (ii).

$$\theta_1=\frac{\pi}{2}-\sin^{-1}\left(\frac{2}{4}\right)=\frac{\pi}{2}-\frac{\pi}{6}=\frac{\pi}{3}$$

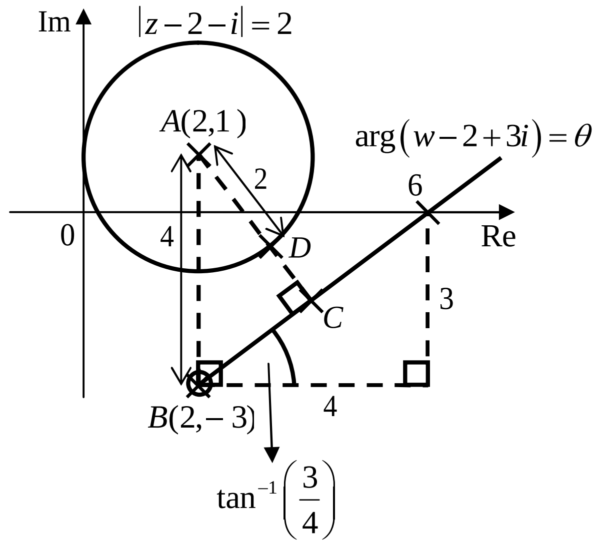
$$\theta_2=\frac{\pi}{2}+\sin^{-1}\left(\frac{2}{4}\right)=\frac{\pi}{2}+\frac{\pi}{6}=\frac{2\pi}{3}$$

$$\theta_1<\theta<\theta_2$$

$$\frac{\pi}{3}<\theta<\frac{2\pi}{3}$$ ■

The line passing through the centre of the circle makes an angle of $\frac{\pi}{2}$ with the positive real axis. Hence, use this angle as a reference, and add/subtract from it.

(ii)(b)



The locus of Q (which represents w) should have gradient

$$\tan\left[\tan^{-1}\left(\frac{3}{4}\right)\right]=\frac{3}{4}$$

Geometrically, the least value of $|z-w|$ is the minimum distance between a point on the locus of P and a point on the locus of Q.

From the diagram, this minimum distance is DC, where $\angle DCB$ and $\angle ACB$ are right angles.

The only useful right-angled triangle is ABC. Since $DC = AC - 2$, aim to find AC first.

Knowing AB and $\angle ABC$, AC can be found using trigonometric ratios on the right-angled triangle ABC.

$AB = 4$

$\angle ABC = \frac{\pi}{2} - \tan^{-1}\left(\frac{3}{4}\right)$

$\sin\angle ABC = \frac{AC}{AB}$

$AC = AB\sin\angle ABC$

$= 4\sin\left[\frac{\pi}{2}-\tan^{-1}\left(\frac{3}{4}\right)\right]$

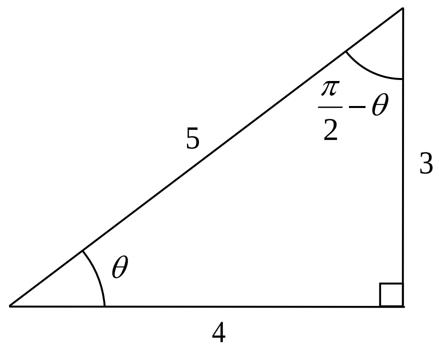
$AC = 4\left(\frac{4}{5}\right) = \frac{16}{5}$

Least value of $|z-w| = DC$

$= AC - 2$

$= \frac{6}{5}$ ■

Let $\theta = \tan^{-1}\left(\frac{3}{4}\right)$.



To calculate $\sin\left(\frac{\pi}{2}-\theta\right)$ in exact form, draw a right-angled triangle involving θ. Since $\tan\theta = \frac{3}{4}$, the side opposite θ and the side adjacent to θ have lengths in the ratio $3:4$.

Permutations & Combinations

Basic Counting Principles

Addition Principle: If there are a ways to do something and b ways to do another thing, and the two things cannot be done simultaneously, then there are $a+b$ total ways to do exactly one of the things.

Multiplication Principle: If there are a ways to do something and b ways to do another thing, then there are $a \times b$ ways to do both things in succession.

Choosing & Permuting

Is the order of objects important?

- Look for keywords in the question, e.g.
  - Important: arrange, order, in a row, around a table
  - Not Important: choose, select, form groups/committees/teams

No →

No. of ways to choose r objects from n distinct objects:

$${}^{n}C_{r} = \frac{n!}{r!(n-r)!}$$

e.g. Choose 2 monitors from a class of 24 students: $n = 24$, $r = 2$

No. of ways $= {}^{24}C_{2}$

Yes → **Can each object be repeated any number of times?**

Yes →

No. of arrangements of r objects (possibly repeated) from n distinct objects:

$$n^r$$

e.g. Form a string of 4 letters using the English alphabet: $n = 26$, $r = 4$, No. of ways $= 26^4$

No → **Are all the objects distinct?**

Yes →

No. of arrangements of r objects (all distinct) from n distinct objects:

$${}^{n}P_{r} = \frac{n!}{(n-r)!}$$

e.g. Choose 1 monitor, 1 treasurer, 1 secretary (all distinct) from a class of 24 students: $n = 24$, $r = 3$,

No. of ways $= {}^{24}P_{3}$

No →

No. of arrangements of n objects, where:

n_1 are of type 1,

n_2 are of type 2, …

n_k are of type k:

$$\frac{n!}{n_1!n_2!\ldots n_k!}$$

e.g. Arrange 2 red balls, 3 green balls, 5 blue balls in a row. Since $2+3+5=10$,

No. of ways $= \frac{10!}{(2!)(3!)(5!)}$

Useful Techniques

- **Circular permutations:** No. of permutations of n distinct objects in a circle $= (n-1)!$, assuming the positions around the circle are identical.

- **Grouping adjacent objects:** If some objects must be adjacent, group them and treat them as a single object. When arranging all the objects/groups, remember to also arrange the grouped objects within each group.

- **Slots:** e.g. If 3 women and 5 men are to be seated in a row such that no 2 women are adjacent, arrange the 5 men first. There are 6 'slots' (denoted |) between and beside the 5 men (denoted M):

  | M | M | M | M | M |

  Then, arrange the 3 women in 3 distinct 'slots'. Hence, the number of ways is $5! \times {}^6P_3$.

- **Complement method:** Use it when direct computation is tedious, especially if key phrases like 'at least one' appear in the question.

  If A is a condition, the number of objects with A equals the total number of objects, minus the number of objects without A:
  $$n(A) = n(\text{total}) - n(A')$$

- **Case-by-case method:** Use it if all else fails. Check that each possibility is counted in one and exactly one case, such that no possibilities are double-counted or missed out.

Example 1 [RJC08/Tutorial 12/Section D/11]

Find the number of permutations of all eight letters of the word BELIEVED in which
(i) there is no restriction;
(ii) the three letter 'E's are next to each other;
(iii) the three letter 'E's are always separated.

	Solution	**Comments**
(i)	$\frac{8!}{3!} = 6720$ ■	The letters in BELIEVED are 3 'E's and 1 each of 'B', 'L', 'I', 'V', 'D'. Since only 'E' is repeated, use $\frac{n!}{n_1!n_2!\ldots n_k!}$ with $n = 8$ and $n_1 = 3$ ($n_2, \ldots, n_8 = 1$).
(ii)	$6! = 720$ ■	Group the 3 'E's as 1 object. Arrange the 6 resulting objects.
(iii)	$5! \times {}^6C_3 = 2400$ ■	Arrange B, L, I, V, D first, for 5! permutations. There are 6 'slots' between and beside these 5 letters: | B | L | I | V | D |. Place the 3 'E's in 3 of these 6 slots. Since the order of the 'E's does not matter, use ${}^6C_3$ instead of ${}^6P_3$.

Example 2 [RJC08/Tutorial 12/Section D/12]

3 boys, 2 girls, and a dog sit at a round table. How many ways can they be arranged if

(i) there is no restriction?
(ii) the dog is to be seated between the 2 girls?
(iii) the dog is to be seated between any 2 boys?

	Solution	**Comments**
(i)	$(6-1)!=120$ ■	Use $(n-1)!$ directly with $n=6$.
(ii)	$(4-1)!\times 2=12$ ■	Group the 2 girls and the dog as 1 object. Arrange the 4 resulting objects, then multiply by 2 as the objects within the group can be arranged in 2 ways. Denoting the girls by G_1 and G_2 and the dog by D, the group can be G_1DG_2 or G_2DG_1 in clockwise order.
(iii)	$(4-1)!\times 3!=36$ ■	Group some 2 boys and the dog as 1 object. Arrange the 4 resulting objects to get $(4-1)!$. Since the 3 boys are distinct, multiply by $3!$ to arrange them.

Example 3 [HCJC01/I/7]

Thirteen cards each bear a single letter from the first thirteen letters A, B, …, L, M of the alphabet. Seven cards are selected at random from the thirteen cards, and the order of selection is not relevant. How many selections can be made if the seven cards

(i) contain at least two but not more than four of the letters A, B, C, D and E?
(ii) contain four but not five consecutive letters, e.g. A, B, C, D, G, H, I?

	Solution	**Comments**
(i)	${}^5C_2\times{}^8C_5+{}^5C_3\times{}^8C_4+{}^5C_4\times{}^8C_3$ $=1580$ ■	Use a case-by-case method. The 7 cards chosen contain: **Case (1)**: exactly 2 of the letters A, B, C, D, E. **Case (2)**: exactly 3 of those letters. **Case (3)**: exactly 4 of those letters. e.g. for Case (1) when there are 2 of those 5 letters, choose those 2 first to get ${}^5C_2$. We are left with 5 letters to choose and 8 letters to choose from, to get ${}^8C_5$.

(ii) $^8C_3 \times 2 + {}^7C_3 \times 8 = 392$ ■

Since there are only 7 cards, they can contain at most 1 string of 4 consecutive letters. Use a case-by-case method.

The 7 cards chosen have no 5 consecutive letters, and:

Case (1):
Contain A, B, C, D or J, K, L, M (2 sub-cases)
Case (2):
Contain any other 4 consecutive letters (8 sub-cases)

For **Case (1)**, after the 4 consecutive letters are fixed, 1 other letter cannot be chosen, since it would form 5 consecutive letters, i.e.:

If A, B, C, D are fixed, E cannot be chosen.
If J, K, L, M are fixed, I cannot be chosen.

We are left with 3 letters to choose and 8 letters to choose from, to get $^8C_3$ for each sub-case.

Case (2) is similar, except after the 4 consecutive letters are fixed, 2 (not 1) other letters cannot be chosen, e.g.:

If B, C, D, E are fixed, A and F cannot be chosen.

Hence, we get $^7C_3$ for each sub-case.

Probability

Definitions

Let A and B be random events.

$P(A \cap B)$ is the probability that both A and B occur.
$P(A \cup B)$ is the probability that only A occurs, only B occurs, or both A and B occur.

- **Conditional probability:** $P(A|B)$ is the conditional probability of A given B. Equivalently, $P(A|B)$ is the probability of A occurring, given that B has occurred.
    - $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$, where $P(B) \neq 0$.

- **Mutually exclusive:** A and B are mutually exclusive if A and B cannot both occur.
    - $P(A \cap B) = 0$
    - $P(A \cup B) = P(A) + P(B)$

- **Independent:** A and B are independent if the occurrence or non-occurrence of one event has no influence on the occurrence or non-occurrence of the other.
    - $P(A|B) = P(A)$
    - $P(B|A) = P(B)$
    - $P(A \cap B) = P(A) \times P(B)$

Useful Techniques

- Use **set diagrams** to illustrate the problem.

- **Complement method:** Use it when direct computation is tedious, especially if key phrases like 'at least one' appear in the question.
$$P(A) = 1 - P(A')$$

- **Case-by-case method:** Use it if all else fails. Check that each possibility is counted in one and exactly one case, such that no possibilities are double-counted or missed out.

- For any 2 random events A and B,
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

  In particular, if A and B are mutually exclusive, then since $P(A \cap B) = 0$,
$$P(A \cup B) = P(A) + P(B)$$

- To find the probability of A occurring given that B has occurred,
  - Calculate $P(A|B)=\dfrac{P(A\cap B)}{P(B)}$, or
  - Consider the reduced sample space given that B has occurred, then calculate $P(A|B)=\dfrac{n(A\cap B)}{n(B)}$, where $n(X)$ is the number of outcomes for X to occur.

- Use concepts from **Permutations & Combinations (P&C)** to count cases.

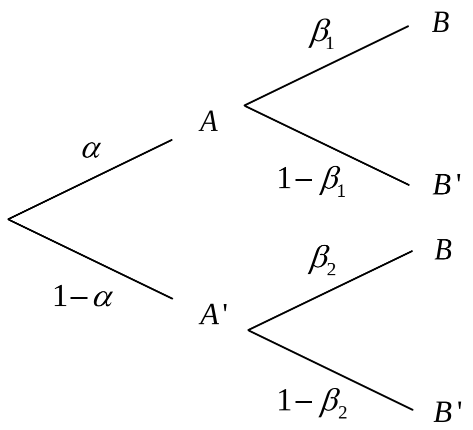
- Use concepts from other topics for alternative approaches to questions.
  - **P&C**, when objects are chosen without replacement.
  - The **Binomial** distribution, when objects are chosen with replacement.

<u>Probability Tree (Tree Diagram)</u>

Use a probability tree when the fixed probabilities or conditional probabilities of 2 or 3 events are given.

Example: $P(A)=\alpha$, $P(B|A)=\beta_1$, and $P(B|A')=\beta_2$.

Probability tree:



- α is the probability of A, while β_1 is the conditional probability of B given A. In general, the probabilities on the first level of the diagram are for only 1 event, while those on the second level are conditional probabilities.

- To find $P(A\cap B)$, $P(A'\cap B)$, $P(A\cap B')$, $P(A'\cap B')$, **multiply** the 2 relevant probabilities.
  e.g. $P(A'\cap B)=P(A')\times P(B|A')=(1-\alpha)\beta_1$

- To find $P(B)$ or $P(B')$, **add** the probabilities of the branches which lead to B or B'.
  e.g. $P(B)=P(A)\times P(B|A)+P(A')\times P(B|A')=\alpha\beta_1+(1-\alpha)\beta_2$

<u>Example 1</u> [D90/II/6]

A bag contains 4 red counters and 6 green counters. 4 counters are drawn at random from the bag, without replacement. Calculate the probability that:

(i) all the counters drawn are green,
(ii) at least 1 counter of each colour is drawn,
(iii) at least 2 green counters are drawn,
(iv) at least 2 green counters are drawn, given that at least 1 counter of each colour is drawn.

State with a reason whether or not the events 'at least 2 green counters are drawn' and 'at least 1 counter of each colour is drawn' are independent.

	Probability approach	**P&C approach**
(i)	$\frac{6}{10}\times\frac{5}{9}\times\frac{4}{8}\times\frac{3}{7}=\frac{1}{14}$ ■	$\frac{{}^{6}C_4}{{}^{10}C_4}=\frac{1}{14}$ ■
	Let G be the number of green counters out of the 4 counters drawn. Then, this is $P(G=4)$. There is no need to multiply by anything else to account for order, since all 4 counters are the same type (green).	Assume the counters are distinct, so that each of the ${}^{10}C_4$ combinations of 4 counters have an equal probability of being chosen. Out of these, ${}^{6}C_4$ combinations are all green.
(ii)	$1-\left(\frac{1}{14}+\frac{4}{10}\times\frac{3}{9}\times\frac{2}{8}\times\frac{1}{7}\right)=\frac{97}{105}$ ■	$1-\left(\frac{1}{14}+\frac{{}^{4}C_4}{{}^{10}C_4}\right)=\frac{97}{105}$ ■
	Using the complement method: $P\left(G=1,2,3\right)$ $=1-P\left(G=0\right)-P(G=4)$ $=1-P\left(G=0\right)-\text{(i)}$	Similar to the Probability approach.
(iii)	$\frac{1}{14}+\frac{97}{105}-\frac{6}{10}\times\frac{4}{9}\times\frac{3}{8}\times\frac{2}{7}\times\frac{4!}{3!}=\frac{37}{42}$ ■	$\frac{1}{14}+\frac{97}{105}-\frac{{}^{6}C_1\times{}^{4}C_3}{{}^{10}C_4}=\frac{37}{42}$ ■
	$P\left(G=2,3,4\right)$ $=P\left(G=4\right)+P\left(G=1,2,3\right)-P\left(G=1\right)$ $=\text{(i)}+\text{(ii)}-P(G=1)$ Multiply by $\frac{4!}{3!}$ when finding $P(G=1)$, to account for order when there are different types of items (green and red).	Again, assume the counters are distinct. There are ${}^{6}C_1$ combinations of 1 green counter, and ${}^{4}C_3$ of 3 red counters, giving ${}^{6}C_1\times{}^{4}C_3$ total combinations of 1 green and 3 red.

	Solution	**Comments**
(iv)	$\left(\frac{37}{42}-\frac{1}{14}\right)\div\frac{97}{105}=\frac{85}{97}$ ■	In probability questions with many parts, look out for ways to use your answers to earlier parts, even though the question may not require it explicitly. We want to find the conditional probability of $G=2,3,4$ given that $G=1,2,3$. Since the intersection of these 2 events is $G=2,3$, using the formula for conditional probability: $\frac{P(G=2,3)}{P(G=1,2,3)}=\frac{P(G=2,3,4)-P(G=4)}{P(G=1,2,3)}=\frac{\text{(iii)}-\text{(i)}}{\text{(ii)}}$
	Let A be the event that at least 2 green counters are drawn. Let B be the event that at least 1 counter of each colour is drawn. Since $P(A\mid B)=\frac{85}{97}$ and $P(A)=\frac{37}{42}$, $P(A\|B)\neq P(B)$, thus A and B are not independent. ■	Since $P(B)$, $P(A)$, and $P(A\mid B)$ are known from parts (ii), (iii), and (iv) respectively, choose the condition $P(A)=P(A\mid B)$ to check if A and B are independent.

Example 2 [J88/II/6]

4 girls, Amanda, Beryl, Claire, and Dorothy, and 3 boys, Edward, Frank, and George, stand in a queue in random order. Find the probability that:

(i) the first 2 in the queue are Amanda and Beryl, in that order,
(ii) either Frank is first or Edward is last (or both),
(iii) no 2 girls stand next to each other,
(iv) all 4 girls stand next to each other,
(v) all 4 girls stand next to each other, given that at least 2 girls stand next to each other.

(i)

Probability approach

$\frac{1}{7}\times\frac{1}{6}=\frac{1}{42}$ ■

Amanda is in position 1 with probability $\frac{1}{7}$, then once there are 6 positions remaining, Beryl is in position 2 with probability $\frac{1}{6}$.

P&C approach

$\frac{5!}{7!}=\frac{1}{42}$ ■

There are $7!$ permutations of the 7 people, each with equal probability. $5!$ of these permutations have Amanda in position 1 and Beryl in position 2.

(ii)

Probability approach

$\frac{1}{7}+\frac{1}{7}-\frac{1}{7}\times\frac{1}{6}=\frac{11}{42}$ ■

Let F be the event that Frank is first, and E be the event that Edward is last. Use $P(F\cup E)=P(F)+P(E)-P(F\cap E)$.

P&C approach

$\frac{6!+6!-5!}{7!}=\frac{11}{42}$ ■

Similar to the probability approach.

(iii)

Probability approach

$\frac{4}{7}\times\frac{3}{6}\times\frac{2}{5}\times\frac{1}{4}=\frac{1}{35}$ ■

The 7 people must be in the order ‘gbgbgbg’, where ‘g’ denotes a girl and ‘b’ a boy.

Position the girls first, then the positions of the boys are fixed and the condition is satisfied.

P&C approach

$\frac{4!\times 3!}{7!}=\frac{1}{35}$ ■

Since we are counting permutations with this approach, we must permute both boys and girls in the ‘gbgbgbg’ order, then divide it by the total of $7!$ permutations.

(iv)

Probability approach

$\frac{4}{7}\times\frac{3}{6}\times\frac{2}{5}\times\frac{1}{4}\times 4=\frac{4}{35}$ ■

This is similar to part (iii), except there are 4 possible positions of ‘gggg’ within or beside ‘bbb’.

P&C approach

$\frac{4!\times 3!\times 4}{7!}=\frac{4}{35}$ ■

Similar to the probability approach.

Solution **Comments**

(v) $\dfrac{\frac{4}{35}}{1-\frac{1}{35}} = \dfrac{2}{17}$ ■

Again, in probability questions with many parts, look out for ways to use your answers to earlier parts, even though the question may not require it explicitly.

Let A be the event that all 4 girls stand next to each other. Let B be the event that at least 2 girls stand next to each other. Then $A \cap B = A$, since B is a subset of A.

$P(B')$ is given by the answer to (iii), since 'at least 2 girls stand next to each other' and 'no 2 girls stand next to each other' are complementary events. Using the formula for conditional probability:

$$P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{P(A)}{P(B)} = \frac{P(A)}{1-P(B')} = \frac{\text{(iv)}}{1-\text{(iii)}}$$

<u>Example 3</u> [N01/II/11a]

Events A and B are such that $P(A) = \frac{1}{3}$, $P(B|A) = \frac{1}{3}$, and $P(A' \cap B') = \frac{1}{6}$.

Find
(i) $P(A \cup B)$,
(ii) $P(B)$.

Solution **Comments**

(i) $P(A \cup B) = 1 - P(A' \cap B') = \frac{5}{6}$ ■

On a Venn diagram, $A \cup B$ is the area within one or both of A and B, and $A' \cap B'$ is the area not within either A or B.

Hence, $P(A \cup B) = 1 - P(A' \cap B')$.

(ii) $P(B)$
$= P(A\cup B) - P(A) + P(A \cap B)$
$= P(A\cup B) - P(A) + P(A)P(B|A)$
$= \frac{5}{6} - \frac{1}{3} + \left(\frac{1}{3}\right)\left(\frac{1}{3}\right)$
$= \frac{11}{18}$ ■

Rewrite the given equations in terms of $P(A)$, $P(B)$, and $P(A\cap B)$ for easier manipulation.

Use
$$P(A\cup B) = P(A) + P(B) - P(A \cap B)$$
then
$$P(B|A) = \frac{P(A\cap B)}{P(A)}$$

Distributions

Expectation

Since the expectation is a summation, you can apply summation rules to it.

Let a and b be constants, and X and Y be random variables.

- $E(a)=a$
- $E(aX)=aE(X)$
- $E(aX\pm b)=aE(X)\pm b$
- $E(aX\pm bY)=aE(X)\pm bE(Y)$

Variance

The variance is always positive. When combined, 2 variances are always added together.

- $\text{Var}(a)=0$, since a constant has no variation.
- $\text{Var}(aX)=a^2\,\text{Var}(X)$
- $\text{Var}(aX\pm b)=a^2\,\text{Var}(X)$
- If X and Y are **independent**, $\text{Var}(aX\pm bY)=a^2\,\text{Var}(X)+b^2\,\text{Var}(Y)$
- Standard deviation $=\sigma=\sqrt{\text{Var}(X)}$
- $\text{Var}(2X)=4\,\text{Var}(X)$, but $\text{Var}(X_1+X_2)=2\,\text{Var}(X)$. **(Example 2)**

Properties of Distributions

- Binomial: $X\sim B(n,p)$
    - $P(X=x)=\binom{n}{x}p^x(1-p)^{n-x}$ (In MF15) for $x=0,1,2,\ldots,n$
    - Mean, $E(X)=np$
    - Variance, $\text{Var}(X)=np(1-p)$
    - Assumptions:
        - n independent trials
        - Constant probability of success
        - Each outcome is either success or failure
        - Occur randomly

- Poisson: $X \sim \text{Po}(\lambda)$
  - $P(X = x) = e^{-\lambda}\dfrac{\lambda^x}{x!}$ (In MF15) for $x = 0,1,2,\ldots$
  - Mean, $E(X) = \lambda$
  - Variance, $\text{Var}(X) = \lambda$
  - Assumptions:
    - Occur independently
    - Uniform rate of occurrence
    - Occur singly
    - Occur randomly

- Normal: $X \sim N(\mu, \sigma^2)$
  - Mean, $E(X) = \mu$
  - Variance, $\text{Var}(X) = \sigma^2$

GC Syntax

If you forget GC syntax, use *CtlgHelp*. For example, to find the syntax for **normalcdf(**,

1. [APPS] → 3:CtlgHelp → [ENTER]
2. [2ND][VARS], then press the down key until **normalcdf(** is highlighted.
3. Press the **[+]** key to display the syntax.

If $X \sim B(n, p)$,

- $P(X = a)$ is given by **binompdf** $(n,\ p,\ a)$
- $P(X \le a)$ is given by **binomcdf** $(n,\ p,\ a)$

If $X \sim \text{Po}(\lambda)$,

- $P(X = a)$ is given by **poissonpdf** $(\lambda,\ a)$
- $P(X \le a)$ is given by **poissoncdf** $(\lambda,\ a)$

If $X \sim N(\mu, \sigma^2)$,

- $P(X < a)$ is given by **normalcdf** $(-E99,\ a,\ \mu,\ \sigma)$
- $P(X > a)$ is given by **normalcdf** $(a,\ E99,\ \mu,\ \sigma)$
- $P(a < X < b)$ is given by **normalcdf** $(a,\ b,\ \mu,\ \sigma)$
- $P(X < a) = P(X \le a)$
- $P(X > a) = P(X \ge a)$
- $P(a < X < b) = P(a \le X < b) = P(a < X \le b) = P(a \le X \le b)$
- For $Z \sim N(0,1)$, the standard normal random variable, μ and σ can be omitted in the GC syntax. The GC will assume that $\mu = 0$ and $\sigma = 1$.

<u>FAQ</u>

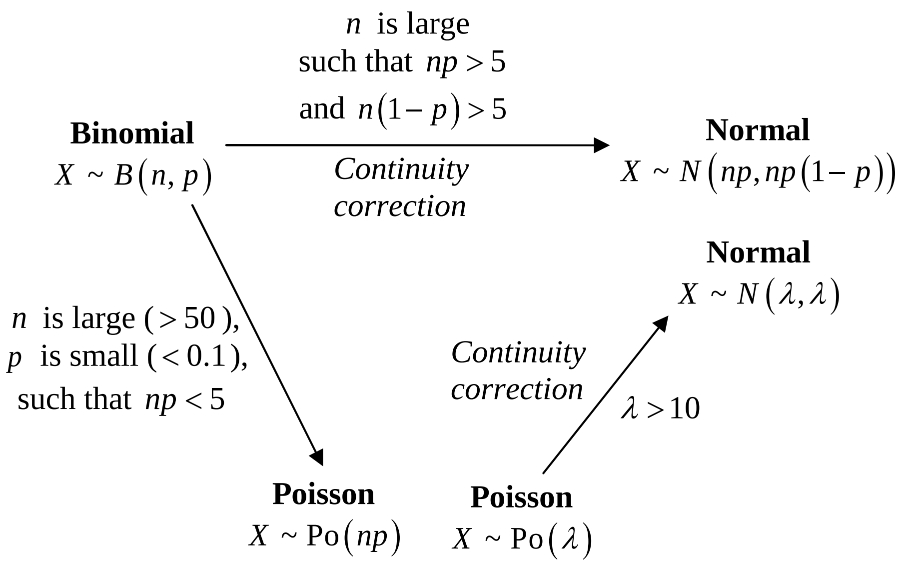
Q: Which distribution does a random variable X follow?

A:
- If X can have **non-integer values**, X is continuous and follows a **normal** distribution.
- If X can have any integer value from 0 onwards with **no upper limit**, X follows a **Poisson** distribution.
- If X can have any integer value from 0 onwards to **an upper limit** n, X follows a **binomial** distribution.

**Q: "Using a suitable approximation,…"
or "Find the approximate number of…"
or "Estimate…"**

A: Show that the original distribution fulfills the conditions for approximation.

Legend: " ⟶ " denotes "approximates to"



If X follows a binomial distribution, i.e. $X \sim B(n, p)$, use the value of p to decide whether to approximate to a Poisson or normal distribution.

- If p is close to 1, approximate X to a Poisson distribution.
- If p is moderate (close to 0.5), approximate X to a normal distribution.
- If p is close to 0, define a new variable Y which has p close to 1, and approximate Y to a Poisson distribution.

Q: Sum random variables.

A: Let X and Y be **independent** random variables.

- If $X \sim \text{Po}(\lambda_1)$ and $Y \sim \text{Po}(\lambda_2)$,
  $X + Y \sim \text{Po}(\lambda_1 + \lambda_2)$
  Subtraction ($X - Y$) is not allowed.

- If $X \sim N(\mu_1, \sigma_1^2)$ and $Y \sim N(\mu_2, \sigma_2^2)$,
  - $X \pm Y \sim N(\mu_1 \pm \mu_2, \sigma_1^2 + \sigma_2^2)$
  - $aX \pm b \sim N(a\mu_1 \pm b, a^2\sigma_1^2)$
  - $aX \pm bY \sim N(a\mu_1 \pm b\mu_2, a^2\sigma_1^2 + b^2\sigma_2^2)$

Q: Solve for multiple unknowns, e.g. μ and σ.

A:
- For Binomial or Poisson, use GC or use the formula for $P(X = x)$.

- For Normal, if μ, σ, or both are unknown, always standardize X to Z.
  1. If $X \sim N(\mu, \sigma^2)$, then let $Z = \dfrac{X - \mu}{\sigma}$.
  2. Then $Z \sim N(0,1)$ and $P(X \leq x) = P\left(Z \leq \dfrac{X - \mu}{\sigma}\right)$.
  3. Since for Z, both mean (0) and standard deviation (1) are known, use **InvNorm** on the GC to find an equation for $\dfrac{X - \mu}{\sigma}$.
  4. Solve simultaneous equations if necessary.

<u>**Example 1**</u> [N01/II/8]

The random variable X has a normal distribution with $P(X > 7.460) = 0.01$ and $P(X < -3.120) = 0.25$. Find the standard deviation of X.

200 independent observations of X are taken.

(i) Using a Poisson approximation, find the probability that at least 197 of these observations are less than 7.460.

(ii) Using a suitable approximation, find the probability that at least 40 of these observations are less than -3.120.

Solution	**Comments**
Let $X \sim N(\mu,\sigma^2)$. Let $Z = \frac{x-\mu}{\sigma}$, then $Z \sim N(0,1)$. $P\left(X > 7.460\right) = P\left(Z > \frac{7.460-\mu}{\sigma}\right) = 0.01$	Since μ and σ are unknown for X, we cannot immediately use InvNorm, and standardization is necessary.
$P\left(Z \le \frac{7.460-\mu}{\sigma}\right) = 0.99$	Once you get an equation in this form, InvNorm can be used as μ and σ are known for Z.
From the GC, $\frac{7.460-\mu}{\sigma} = 2.3263$ (5 s.f.)	Key in **InvNorm**(0.99). If you do not specify μ and σ, the GC will assume $\mu = 0$ and $\sigma = 1$, as with the standard normal distribution.
(working omitted) From the GC, $\frac{-3.120-\mu}{\sigma} = -0.67449$ (5 s.f.)	Similarly, standardize and use InvNorm for the other equation from the question, $P\left(X < -3.120\right) = 0.25$.
$\begin{cases} \mu + 2.3263\sigma = 7.460 \\ \mu - 0.67449\sigma = -3.120 \end{cases}$ From the GC, $\sigma = 3.53$ (3 s.f.) ■	Write the system of linear equations and use PlySmlt2 on the GC to solve for μ and σ.
(i) Let Y be the number of observations not less than 7.460 out of 200 observations. Then $Y \sim B(200, 0.01)$.	We define Y with "**not less than**" rather than "less than", because p must be small for a Poisson approximation.
Since $n = 200$ is large (>50) and $p = 0.01$ is small such that $np = 2 < 5$, $Y \sim Po(2)$ approximately.	Remember to write "out of 200 observations".
$P\left(200 - Y \ge 197\right)$ $= P\left(Y \le 3\right) = 0.857$ (3 s.f.) ■	$200 - Y$ is the number of observations **less than 7.460**, as required.
(ii) Let W be the number of observations less than -3.120 out of 200 observations. Then $W \sim B(200, 0.25)$.	Z is reserved for the variable with the standard normal distribution, so define the variable as W instead.

Since $n = 200$ is large such that $np = 50 > 5$ and $n(1-p) = 150 > 5$, $W \sim N(50, 37.5)$ approximately.

To approximate a binomial variable with moderate p e.g. 0.25, use a normal approximation. If p is extreme e.g. 0.01, use a Poisson approximation.

$P(W \geq 40) = P(W > 39.5)$
(by continuity correction)
$= 0.957$ (3 s.f.) ■

Remember to key in $\sigma = \sqrt{37.5}$ for normalcdf on the GC, not $\sigma^2 = 37.5$.

Example 2 [AJC05/II/27]

The weights of large eggs are normally distributed with mean 65 grams and standard deviation 4 grams. The weights of standard eggs are normally distributed with mean 50 grams and standard deviation 3 grams.

(a) One large egg and one standard egg are chosen at random. Find the probability that the weight of the standard egg is more than $\frac{4}{5}$ the weight of the large egg.

(b) Standard eggs are sold in packs of 12 while large eggs are sold in packs of 5.

(i) Find the probability that the weight of a pack of standard eggs differs from twice the weight of a pack of large eggs by at most 5 grams.

(ii) A customer bought two packs of large eggs. Find the value of *m* (to the nearest 0.1 gram) if there is a probability of 0.03 that the average weight of the eggs in one pack is more than the average weight of the eggs in the other pack by at least *m* grams.

Solution | **Comments**

(a) Let X represent the weight of a large egg, in grams.
Let Y represent the weight of a standard egg, in grams.
Then $X \sim N(65, 4^2)$ and $Y \sim N(50, 3^2)$

Required probability is $P\left(Y > \frac{4}{5}X\right)$

$E\left(Y - \frac{4}{5}X\right) = 50 - \frac{4}{5}(65) = -2$

$\text{Var}\left(Y - \frac{4}{5}X\right) = 3^2 + \frac{16}{25}(4)^2 = 19.24$

$\therefore Y - \frac{4}{5}X \sim N(-2, 19.24)$

$\Rightarrow P\left(Y > \frac{4}{5}X\right) = P\left(Y - \frac{4}{5}X > 0\right) = 0.324$ (3 s.f.) ■

(b)(i) Required probability

$= P\left(\left|Y_1 + Y_2 + ... + Y_{12} - 2(X_1 + ... + X_5)\right| < 5\right)$

$E\left(Y_1 + ... + Y_{12} - 2(X_1 + ... + X_5)\right)$

$= 50 \times 12 - 2(5 \times 65)$

$= -50$

$Var\left(Y_1 + ... + Y_{12} - 2(X_1 + ... + X_5)\right)$

$= 12 \times 3^2 + 4(5 \times 16)$

$= 428$

$\therefore Y_1 + ... + Y_{12} - 2(X_1 + ... + X_5) \sim N(-50, 428)$

$P\left(\left|Y_1 + Y_2 + ... + Y_{12} - 2(X_1 + ... + X_5)\right| < 5\right)$

$= P\left(-5 < Y_1 + ... + Y_{12} - 2(X_1 + ... + X_5) < 5\right)$

$= 0.0109$ (3 s.f.) ■

A pack of standard eggs contains 12 different eggs, hence each egg is an independent observation of Y.

$\frac{X_1 + ... + X_5}{5} - \frac{X_6 + ... + X_{10}}{5} \sim N(0, 6.4)$

Let $Q = \frac{X_1 + ... + X_5}{5} - \frac{X_6 + ... + X_{10}}{5}$

$P(|Q| \geq m) = 0.03 \Rightarrow P(Q < -m) = 0.015$

From G.C., $m = 5.5g$ (nearest 0.1g) ■

Even though one pack is more than the other, use the modulus sign because either pack could be the heavier one.

<u>Example 3</u> [NJC05/II/30]

A Geography student is studying the distribution of bullfrogs in a large rural field where there is an average of 500 bullfrogs per 400 km^2. One part of the field is identified and divided into 50 equal squares, each with sides measuring 2 km. The student wishes to model the distribution of bullfrogs in each square by using the Poisson distribution with mean λ.

(i) State the value of λ and one assumption made in using the Poisson distribution.

(ii) Four squares are selected at random. Find the probability that two of these squares contain no bullfrogs and each of the other two squares contains at least 2 bullfrogs.

(iii) Using a suitable approximation, find the probability that, out of 50 squares, there are at least 35 squares that contain at most 5 bullfrogs each.

(iv) Determine the approximate probability that the total number of bullfrogs in 10 randomly selected squares is between 40 and 60 (not inclusive).

(v) The student suggests using the same model on another rural field in another country. Comment on the suitability of the model in this situation.

Solution

(i) Area of a square is 4km^2.

Total squares in field $= \dfrac{400}{4} = 100$.

$\lambda = \dfrac{500}{100} = 5$ ■

Assumption: The event that a bullfrog lies in a square is independent of other bullfrogs. ■

(ii) Let X be number of bullfrogs in a square.

Then $X \sim Po(5)$

Required probability

$= [P(X=0)]^2 (1 - P(X \le 1))^2 \times \dfrac{4!}{2!2!}$

$= 0.000251$ (3 s.f.)

(iii) $P(X \le 5) = 0.61596$

Let Y be the number of squares that contain at most 5 bullfrogs, out of 50.

$Y \sim B(50, 0.61596)$

Since $n = 50$ is large,

$np = 30.8 > 5, n(1-p) = 19.2 > 5,$

$Y \sim N(30.798, 11.828)$ approximately.

$P(Y \ge 35)$

$= P(Y > 34.5)$ (by continuity correction)

$= 0.141$ (3 s.f.) ■

Comments

Do not write the "probability" that a bullfrog lies in a square is independent. Use the word "event".

<u>Alternative assumption</u>

Average rate of occurrence of bullfrogs in the field is constant.

Use P & C:

There are $\dfrac{4!}{2!2!}$ permutations of 4 objects where 2 objects are of type A and the other 2 are of type B.

If $np > 5$, try approximating binomial to normal. Otherwise, try binomial to Poisson, since these are the only two possible cases.

(iv) Let Q denote number of bullfrogs in 10 squares.

Then $Q \sim Po(50)$

Since $\lambda = 50 > 10$,

$Q \sim N(50, 50)$ approximately.

$P(40 < Q < 60)$

$= P(40.5 < Q < 59.5)$ (by continuity correction)

$= 0.821$ (3 s.f.) ■

(v) The number of bullfrogs may not be randomly distributed in another country. Therefore, the model may not be suitable. ■

Sampling & Testing

Sampling Methods

Example: Say we want to select a sample of 50 workers from a company with 200 workers. 40 workers are under 30 years old, 100 are 30 to 40 years old, and 60 are above 40 years old.

<table>
<tr><td></td><td>Simple random sampling</td><td>Systematic sampling</td></tr>
<tr><td></td><td>Random</td><td>Random</td></tr>
<tr><td rowspan="2">Method</td><td colspan="2">Assign each worker a number from 1 to 200.</td></tr>
<tr><td>Select a sample of 50 workers using a list of random numbers.</td><td>As $\frac{200}{50} = 4$, randomly select one worker from the first 4. If say the 3rd worker is chosen, include every 4th worker from the 3rd worker in the sample, i.e. the 3rd, 7th, 11th, …, 199th workers.</td></tr>
<tr><td>Pros</td><td></td><td>• Even spread over population</td></tr>
<tr><td>Cons</td><td>• Requires sampling frame
• Cannot replace unavailable sampling units
• Time-consuming to locate selected units</td><td>• Requires sampling frame
• Cannot replace unavailable sampling units
• Biased if members of the population have a periodic or cyclic pattern</td></tr>
</table>

<table>
<tr><td></td><td>Stratified sampling</td><td>Quota sampling</td></tr>
<tr><td></td><td>Random</td><td>Non-random</td></tr>
<tr><td>Method</td><td>Select random samples from the age groups with sample size proportional to the relative size of each age group, i.e.:
<table><tr><td>Age</td><td>Sample size</td></tr><tr><td><30</td><td>20</td></tr><tr><td>30 – 40</td><td>50</td></tr><tr><td>>40</td><td>30</td></tr></table></td><td>Same as stratified, except:
• Sample size need not be (but can be) proportional
• Non-random samples are drawn from each age group</td></tr>
<tr><td>Pros</td><td>• Likely to give more representative sample</td><td>• Lower cost
• Less time-consuming
• Does not require sampling frame</td></tr>
<tr><td>Cons</td><td>• Requires sampling frame
• Cannot replace unavailable sampling units
• Strata may not be clearly defined</td><td>• Non-random
• Likely to give less representative sample
• Biased as researcher may just pick members who are easier to collect data from</td></tr>
</table>

FAQ (Sampling)

Questions on sampling methods	Use the context of the question in your answer. • Identify type of sampling method used • Explain why: o Method is inadequate: list cons o Method is biased/unbiased o Sample is likely to be representative or not • Suggest a better method if required • Describe your suggested method
Given $\sum x$ and $\sum x^2$, or $\sum (x-\bar{x})^2$, • Find unbiased estimate $\bar{x}$ of population mean μ • Find unbiased estimate s^2 of population variance σ^2	$\bar{x}=\frac{1}{n}\sum x$ $s^2=\frac{1}{n-1}\left(\sum x^2-\frac{\left(\sum x\right)^2}{n}\right)=\frac{1}{n-1}\sum(x-\bar{x})^2$
Find $\bar{x}$ and s^2, given $\sum(x+a)$ and $\sum(x+b)^2$.	• Let $y=x+b$. • $\sum y$ $=\sum(x+a)+\sum b-\sum a$ $=\sum(x+a)+n(b-a)$ • $\sum y^2=\sum(x+b)^2$ • Find $\bar{y}$ and s_y^2 (see above). • $\bar{x}=\bar{y}-b$, $s_x^2=s_y^2$
Find $\bar{x}$ and s^2, given a list of data.	GC gives s_x, square the value to get s_x^2. • Individual data o Values in L_1 o Key in: 1-Var Stats L_1 • Grouped data o Values in L_1, frequencies in L_2 o Key in: 1-Var Stats L_1, L_2

Testing

Do this:	**Only if:**
Use t-test instead of z-test	• sample is small ($n<50$), and • σ^2 is unknown
Use s^2 instead of σ^2	• σ^2 is unknown
Use Central Limit Theorem	• sample is large ($n\geq 50$), and • population distribution is unknown
Assume population distribution is normal	• sample is small ($n<50$), and • population distribution is unknown

If the sample is large, population is normal, and σ^2 is unknown, Central Limit Theorem is not required. However, since s^2 must be used as an unbiased estimate of σ^2, write "approximately" after the distribution of $\bar{X}$.

FAQ (Testing)

Explain what is meant by 'a α% significance level' in the context of the question	A α % significance level means that the probability of concluding that H_0 is rejected given that H_0 is true is $\frac{\alpha}{100}$. **Example:** A 5% significance level means that the probability of concluding that the mean mass of a chicken is greater than 1.75kg when it is actually 1.75kg is 0.05.
State the assumptions made, if any	• If the sample is large ($n \geq 50$): o Central Limit Theorem holds, hence the population follows a normal distribution approximately, and no assumptions about its distribution are needed. • If the sample is small ($n < 50$): o Assume the distribution of the population is normal
Compare outcomes of z-test and t-test, given same data	Since the graph of the t-distribution has "thicker tails" than that of the z-distribution about μ_0, given the same significance level, the t-test would be lead to a conclusion of H_0 being rejected for more values of $\bar{x}$.
Compare effect of change in significance level, sample mean, or population mean on outcome	• If $p-value = P\left(\bar{X} \leq \bar{x}\right) = P\left(Z \leq \frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}}\right)$, o If $\frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}}$ increases, $p-value$ increases • If $p-value = P\left(\bar{X} \geq \bar{x}\right) = P\left(Z \geq \frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}}\right)$, o If $\frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}}$ increases, $p-value$ decreases

Find the range of α, $\bar{x}$, μ_0, s, or n such that H_0 is rejected (or not rejected) at a $\alpha\%$ significance level.	• Find an expression for the p-value o 1-tailed: $P\left(\bar{X} \le \bar{x}\right)$ or $P\left(\bar{X} \ge \bar{x}\right)$ o 2-tailed: ▪ If $\bar{x} \le \mu_0$, $2P\left(\bar{X} \le \bar{x}\right)$ ▪ If $\bar{x} \ge \mu_0$, $2P\left(\bar{X} \ge \bar{x}\right)$ • State the condition on the p-value for the outcome (For e.g. a 5% significance level, $\alpha - value = 0.05$) o If H_0 is rejected, $p - value \le \alpha - value$ o If H_0 is not rejected, $p - value > \alpha - value$ • Standardize, only if the mean μ_0 or variance $\frac{s^2}{n}$ of $\bar{X}$ is unknown, i.e. when finding the range of μ_0, s, or n • Use **InvNorm** to find the range of α, $\bar{x}$, or μ_0 that satisfies the condition on the p-value • Conclude in the context of the question

<u>Example</u> [TJC07/Prelim/II/9 (Modified)]

The manufacturer of the ‘Genius’ brand of cigarettes claimed that the average tar content of a ‘Genius’ cigarette is not more than 14mg per cigarette (mg/cig). A surveyor from a rival company took a random sample of 8 ‘Genius’ cigarettes and lab measurements of the tar content yielded 14.2, 14.5, 13.9, 14.1, 14.0, 14.1, 14.2, 14.4 mg/cig. Will he be able to reject the manufacturer’s claim at 5% level of significance? State any necessary assumption made by the surveyor in carrying out the test.

To confirm his findings, the surveyor took another random sample of 50 ‘Genius’ cigarettes and measured the tar content x in mg/cig. The following results were obtained: $\sum\left(x-10\right) = 208.2515$, $\sum\left(x-14\right)^2 = 5.2455$.

Based on this sample,

(i) Find unbiased estimates of the population mean and variance of the tar content in mg/cig.

(ii) Find the range of values for the average tar content per cigarette that the manufacturer should claim in order that it will not be rejected as an underestimate at the 1% level of significance if a z-test is carried out.

Solution

Let X be the tar content of a cigarette in mg/cig.

Since $n=8$ is small, the surveyor must assume that X follows a normal distribution.

$H_0 : \mu = 14$ vs $H_1 : \mu > 14$

Perform a 1-tailed test at the 5% significance level.

Under H_0, $T = \dfrac{\bar{X} - \mu_0}{s/\sqrt{n}} \sim t(n-1)$,
where $\bar{x} = 14.175$, $\mu_0 = 14$,
$s = \sqrt{0.19821}$, $n = 8$.

Using a t-test, $p-value = 0.0206$ (3 s.f.)

Since $p-value = 0.0206 < 0.05$, the surveyor will be able to reject H_0 and conclude that there is sufficient evidence at the 5% significance level that $\bar{X} > 14$ and the manufacturer's claim is false. ■

Comments

Always define X if the question does not. Specify its units, if any (mg/cig).

Since the sample is small and the distribution of the population is unknown, it must be assumed to be normal.

1. State the null hypothesis H_0 and the alternative hypothesis H_1.

2. Write down 1-tailed or 2-tailed test, and the level of significance $\alpha\%$.

3. Decide on the test statistic (t or z) to be used and determine its distribution.

$\bar{x}$ and s^2 will appear on the GC screen with the p-value, so we do not need to calculate them.

$T = \dfrac{\bar{X} - \mu_0}{s/\sqrt{n}}$, not $\dfrac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$, since σ is unknown and we estimated it with s.

$\bar{X}$ is a random variable representing the sample mean, which can have many values. $\bar{x}$ is the mean of a particular sample, usually given in the question.

4. Use the GC to calculate p-value.

5. Conclude in the context of the question.

(i) Let $y = x - 14$.

$\sum y = \sum (x-14)$

$= \sum (x-10) - 50(4) = 8.2515$

$\sum y^2 = \sum (x-14)^2 = 5.2455$

$\bar{x} = \bar{y} + 14 = \frac{8.2515}{50} + 14$

$= 14.165$ (5 s.f.) $= 14.2$ (3 s.f.) ■

$s_x^2 = \frac{1}{50-1}\left(5.2455 - \frac{8.2515^2}{50}\right)$

$= 0.079260$ (5 s.f.) $= 0.0793$ (3 s.f.) ■

Always let y be the expression which is squared in the summary statistics, e.g. let $y = x - 14$ when $\sum (x-14)^2$ is given.

Give values which will be used in later parts of the question to 5 s.f. first, then to 3 s.f.

(ii) $H_0 : \mu = \mu_0$ vs $H_1 : \mu > \mu_0$.

Perform a 1-tailed test at the 1% significance level.

Under H_0, by the Central Limit Theorem since $n = 50$ is large,

$\bar{X} \sim N\left(\mu_0, \frac{s^2}{n}\right)$ approximately, where

$\bar{x} = 14.165$, $s = \sqrt{0.079260}$, $n = 50$.

Let $Z = \frac{\bar{X} - \mu_0}{s/\sqrt{n}}$. Then $Z \sim N(0,1)$.

If H_0 is not rejected, $p-value > 0.01$.

$P\left(\bar{X} \geq 14.165\right) = P\left(Z \geq \frac{\bar{x} - \mu_0}{s/\sqrt{n}}\right) > 0.01$

$P\left(Z < \frac{\bar{x} - \mu_0}{s/\sqrt{n}}\right) < 0.99$

From the GC,

$\frac{14.165 - \mu_0}{\sqrt{0.079260/50}} < 2.3263$ (5 s.f.)

Solving, $\mu_0 > 14.1$ (3 s.f.) ■

Translate the question into mathematical terms: What is the range of μ_0 such that H_0 is not rejected when a z-test is carried out at the 1% significance level?

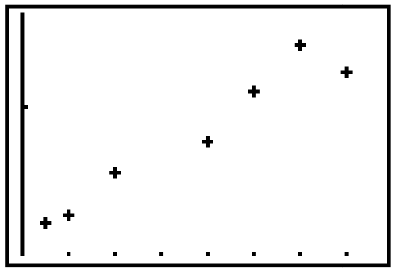
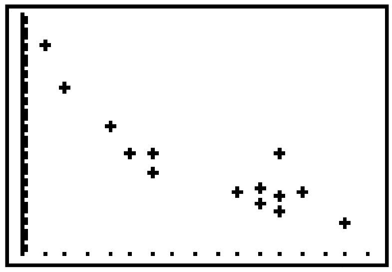
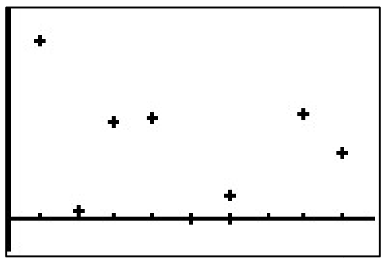
The manufacturer's claim is $H_0 : \mu = \mu_0$, and if it were "rejected as an underestimate, $H_1 : \mu > \mu_0$ would be true.

Since μ_0 is unknown for $\bar{X}$, we cannot immediately use InvNorm, and standardization is necessary.

Once you get an equation in this form, InvNorm can be used as μ and σ are known for Z.

Correlation & Regression

<u>Product Moment Correlation Coefficient (PMCC)</u>

r > 0 ➔ Positive Linear Correlation	
r < 0 ➔ Negative linear correlation	
r ≈ 0 ➔No linear correlation (Note that no linear correlation does not imply no relation)	

- The closer the absolute value of r is to 1, the **<u>stronger</u>** the linear correlation between 2 variables.

<u>Limitations of PMCC</u>

- Does not necessarily prove that there is a linear relationship between two variables
  - e.g. there may be an outlier
- Unable to prove any non-linear relationship between variables using the coefficient
  - e.g. points on a parabola can be chosen to give a PPMC of close to zero

Therefore, use BOTH the scatter diagram and the *r*-value together when making judgements/statements/comparisons.

Relevant Formulae (In MF15)

Estimated Product Moment Correlation Coefficient
$r = \dfrac{\sum (x-\bar{x})(y-\bar{y})}{\sqrt{\left\{\sum (x-\bar{x})^2\right\}\left\{\sum (y-\bar{y})^2\right\}}} = \dfrac{\sum xy - \dfrac{\sum x \sum y}{n}}{\sqrt{\left(\sum x^2 - \dfrac{\left(\sum x\right)^2}{n}\right)\left(\sum y^2 - \dfrac{\left(\sum y\right)^2}{n}\right)}}$

Estimated regression line of y on x
$y - \bar{y} = b(x - \bar{x})$ $b = \dfrac{\sum (x-\bar{x})(y-\bar{y})}{\sum (x-\bar{x})^2}$

Although not in MF15, the formula for the regression line of x on y can be obtained by swapping x with y in the above formula for the regression line of y on x)

Linear Regression

The least square regression line is the line which minimises the sum of squares of the residuals. (Note that the a residual is the difference between the observed & predicted values.)

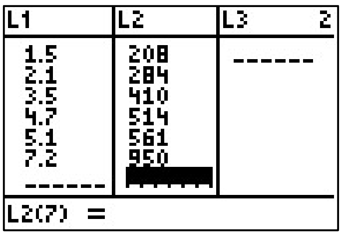
Scatter Diagrams

- Label each axis with the variable and its units.
- Ensure the relative position of the data points is correct.
- Display the origin at the correct relative position. If the data is far away from the origin, use a jagged line to indicate an omitted range of values.

GC Tips & FAQ

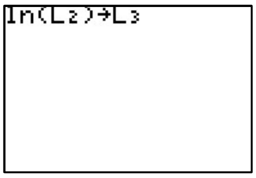
1. How do I key in the data set into the GC?

Key STAT ➔EDIT ➔ Type in your data



2. **When a question requires me to plot ln *y* against *x*, but it only provides data set for *x* and *y*, what can I do other than manually evaluating each value of ln *y*?**

   For example, if the data given is a list of values of *y*, but the question requires you to plot ln *y* against *x*, you can do so using GC. For example, *y* values are in L2. Key all your values into the lists, with *x* values in L1, *y* values in L2. Then, key ln(L2) ➔ STO ➔ L3



3. **What is the significance of the intersection point of regression lines of *x* on *y* and *y* on *x*?**

   The intersection point of regression lines of *x* on *y* and *y* on *x* is $(\bar{x}, \bar{y})$.

4. **How do you assess whether an estimate is reliable?**

- First, we look at the *r*-value. If it is close to 1 or – 1, then the estimate would be reliable.
- Next, we look at whether it is an interpolation or an extrapolation. Assuming *r* is close to 1 or – 1, an interpolation would be very reliable.
- In the case of an extrapolation, we would look the amount of extrapolation. If it is close to the experimental range of values, then that estimate is reliable. Otherwise, it is not reliable.

  Note that you should use the scatter plot in line with the *r*-value.

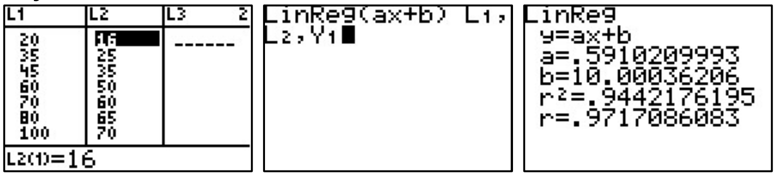
Example 1 [SAJC08/Prelim/II/6]

Each of a random sample of 10 students are asked about the average number of minutes spent on doing mathematics tutorials in a week (*x*), and their percentage score for the mathematics final examination (*y*). The results are tabulated below:

x	20	35	45	60	70	80	100	110	120	140
y	16	25	35	50	60	65	70	75	80	85

(i) Find the equation of the regression line of *y* on *x*.

The question is indirectly saying that y is the dependent variable and x is dependent.
Key all the values into the lists in GC. Remember to turn on the diagnostic function.



From G.C.,
$y = 0.591x + 10.0\,(3s.f.)$
[Whether you use LinReg(ax+b) or LinReg(a+bx), linear regression should be done with L1, L2 where L1 is the independent variable and L2 is the dependent variable.]

(ii) Find the linear product moment correlation coefficient between *y* and *x*, and comment on the relationship between *x* and *y*.

Use the G.C values calculated earlier
$r \approx 0.972\,(3s.f.)$

(iii) Use the appropriate regression line to estimate the percentage score of a student who spends 10 minutes doing mathematics tutorial in a week. Comment on the reliability of the estimate.

$y = 0.591x + 10.0\,(3s.f.)$
Using G.C, $y \approx 15.9$
However, the estimate is not reliable as 10 minutes is well outside the given range of *x* values so the approximation would be unreliable.

<u>**Example 2**</u> [HCI08/Prelim/II/8 (Modified)]

A teacher decided to investigate the association between students' performances in Mathematics and Physics. She selected 5 students at random and recorded their scores, *x* and *y*, in Mathematics and Physics tests respectively. She found that, for this set of data, the equation of the regression line of *y* on *x* was $y = 18.5 + 0.1x$ and the equation of the regression line of *x* on *y* was $x = 16.6 + 0.4y$.

Show that $\sum x = 125$ and $\sum y = 105$.

$$\begin{cases} 18.5 + 0.1\bar{x} = \bar{y} \\ 16.6 + 0.4\bar{y} = \bar{x} \end{cases}$$

$\bar{x} = 25$ and $\bar{y} = 21$

$\therefore \sum x = 5 \times 25 = 125$ and $\sum y = 5 \times 21 = 105$

Remember that the intercept of the regression lines (y on x and x on y) is $(\bar{x}, \bar{y})$. Combination of concepts from sampling

(i) **The Physics score of a sixth student was mislaid but his Mathematics score was known to be 26.**

Use the appropriate line of regression to estimate this student's Physics score, and give a reason for the use of the chosen equation.

$y = 18.5 + 0.1x$

When x = 26, y = 21(nearest integer)

Since x is given, y is the dependent variable and therefore we use the regression line of y on x.

(iii) **Comment on the reliability of your estimate.**

The estimate is not reliable because $|r|$ is small, i.e. it is close to 0, hence the linear correlation between x and y is very weak, indicating that the linear model is not suitable.

<u>Example 3</u> [NYJC08/Prelim/II/11]

A student wishes to determine the relationship between the length of a pendulum, l, and the corresponding period, T. After conducting the experiment, he obtained the following set of data:

l/cm	150	135	120	105	90	75	60	45	30	15
T/s	2.45	2.31	2.22	2.07	1.91	1.74	1.56	1.35	1.10	0.779

(i) Obtain the scatter plot of this set of data. [2]

(ii) The student proposes two models.

$$A: T = a + b\ln(l)$$

$$B: T^2 = a + bl$$

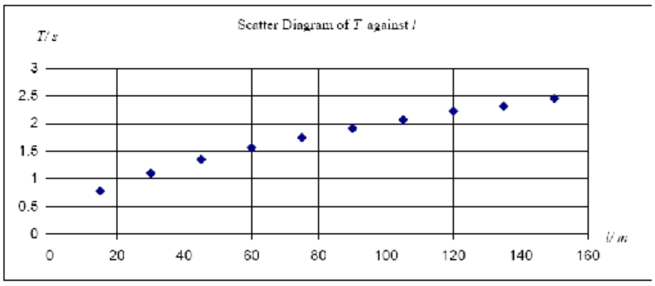
Calculate the product moment correlation coefficients for both models, giving your answer to 4 decimal places. Determine which model is more appropriate. [3]

(iii) Using the model determined in part (ii), estimate the value of l when T = 3s to 1 decimal place. Comment on the suitability of this method. [3]

(iv) Find a value of l and its corresponding value of T such that the equation of the regression line for the chosen model will remain the same after the addition of this pair of values. [3]

Solution | **Comments**

(i)



Note that your scatter plot must be *to scale*. This means that your points should be relatively correct to each other.

Axis should be labelled correctly.

(ii) Using a G.C.,
r for model A is 0.9871. (4 d.p.)
r for model B is 0.9996. (4 d.p.)

Since the value of r for model B is closer to 1 as compared to model A, model B is more appropriate.

Be careful as question asks for 4 decimal places.

For appropriateness of model, look at r values.

(iii) Since T is given, we should use the regression line of l on T^2.

From G.C., the value of l is 223.9 (1d.p.)

Although r is close to 1, the value of T = 3s lie well outside of the range of the T values used to plot the regression line. Hence it is not a good estimate of l.

Even though it is in the Physics mindset that the period depends on the length of the pendulum, it is NOT the regression line of T^2 on l.

Be aware that model B uses T^2 instead of T.

(iv) The values to be used are $\bar{l}$ and $\overline{T^2}$.

$$l = \bar{l} = 82.5 \text{ and } \overline{T^2} = 3.3301 \; (5 \text{ s.f.})$$

$$T = \bar{T} = 1.82 \, (3 \text{ s.f.})$$

Use G.C. to evaluate the values.

Note that you can plot the regression lines for both l on T^2 and T^2 on l. The intersection of the two lines will obviously be the point you are looking for.